1 General Convolution

Let $\mathcal{X}$ be an ∞-topos, and let $H \in \mathrm{Grp}(\mathcal{X}) \subset \mathrm{Mon}(\mathcal{X}) \simeq \mathrm{Alg}(\mathcal{X}^\times)$ be a group-object and $X \in \mathrm{Alg}({}_H\mathrm{BMod}_H(\mathcal{X}^\times)^{\otimes_H})$ be a monoid-object in $\mathcal{X}$ with an H-H-equivariant multiplication for some action of H from both the left and right. If $D : \mathrm{Corr}(\mathcal{X})^\otimes \to \mathrm{Cat}^\times$ is a six-functor formalism (or even a three-functor formalism), then the category $D(H\backslash X/H)$ admits a monoidal structure different from the usual tensor product, called the *convolution product*, denoted $\star$. The multiplication is explicitly given by "pull-push" along the span

$$\begin{array}{ccccc} & & H\backslash(X \otimes_H X)/H & & \\ & \overset{p}{\swarrow} & & \overset{m}{\searrow} & \\ H\backslash X/H \times H\backslash X/H & & & & H\backslash X/H. \end{array}$$

That is,

$$\mathcal{F} \star \mathcal{G} \simeq m_! p^*(\mathcal{F} \boxtimes \mathcal{G})$$

In this section, we will make this into a fully coherent monoidal structure, promoting $D(H\backslash X/H)$ to a monoidal ∞-category $D(H\backslash X/H)^\star \in \mathrm{Mon} \simeq \mathrm{Alg}(\mathrm{Cat}^\times)$ of "H-H-equivariant sheaves on X."

An example of interest is the case $\mathcal{X} \simeq \mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}_k)$ and the monoid-object is the loop group LG, and the group-object is the positive loop group L^+G acting by multiplication on both the left and right. This yields a convolution product on the double-fpqc-quotient $\mathrm{Hck}_G = L^+G\backslash LG/L^+G$, or equivalently, on L^+G-equivariant constructible sheaves on the affine Grassmannian $\mathrm{Gr}_G = LG/L^+G$, with an analogous version for the Beilinson-Drinfeld affine Grassmannian.

1.1 Double categories and the category of correspondences

The main technical tool is the fully faithful functor from *orthogonal factorization systems*[1] to *double categories*[2],

$$\mathrm{OFS} \xrightarrow{\mathrm{Fact}} \mathrm{DCat}$$

constructed and proven in [Jur25, Theorem A]. Explicitly, if $\mathcal{A}^\dagger \in \mathrm{OFS} \subset \mathrm{Fun}(\Lambda^2_2, \mathrm{Cat})$ is an orthogonal factorization system, then $\mathrm{Fact}(\mathcal{A}^\dagger)$ is the bisimplicial object given by

$$\mathrm{Fact}(\mathcal{A}^\dagger)([m],[n]) = \mathrm{Map}([m]\overline{\times}[n], \mathcal{A}^\dagger),$$

where $[m]\overline{\times}[n]$ denotes the orthogonal factorization system $([m] \times [n]^\simeq \to [m] \times [n] \leftarrow [m]^\simeq \times [n])$.

Theorem 1.1.1. If $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ is a lax monoidal map of monoidal ∞-categories whose underlying functor $G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$, then F promotes to an oplax monoidal functor $F : \mathcal{C}_\otimes \to \mathcal{D}_\otimes$.

Theorem 1.1.2. If $\mathcal{C}^\otimes$ is symmetric monoidal and $f : A \to B$ is a map in $\in \mathrm{Alg}(\mathcal{C}^\otimes)$, then there is an adjunction of ∞-categories

$${}_A\mathrm{BMod}_A(\mathcal{C}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} {}_B\mathrm{BMod}_B(\mathcal{C})$$

where f_* is the identity on objects and $f^*(M) = B \otimes_A M \otimes_A B$. Furthermore, f_* promotes to a lax monoidal map, and (therefore) f^* promotes to an oplax monoidal map.

From these results, we find

[1] An ∞-category $\mathcal{C}$ and two subcategories $\mathcal{C}_{\mathrm{eg}}, \mathcal{C}_{\mathrm{in}} \subset \mathcal{C}$ containing every equivalence such that the map

$$(\mathcal{C}_{\mathrm{eg}})^\simeq \times_{\mathcal{C}^\simeq} (\mathcal{C}_{\mathrm{in}})^\simeq \to (\mathcal{C})^\simeq, \quad (f,g) \mapsto g \circ f$$

is an equivalence. Informally, this just means that every map in $\mathcal{C}$ factors uniquely as a map in $\mathcal{C}_{\mathrm{eg}}$ followed by a map in $\mathcal{C}_{\mathrm{in}}$.

[2] A bisimplicial object in An satisfying the Segal condition in each coordinate. Informally, this is a grid where the first category lies horizontally, and the second lies vertically.

Corollary 1.1.3. If $\mathcal{X}$ is an ∞-topos, and $A \in \mathrm{Mon}(\mathcal{X}) \simeq \mathrm{Alg}(\mathcal{X}^{\times})$ is a monoid-object, then there is an oplax monoidal functor

$$ {}_A\mathrm{BMod}_A(\mathcal{X}^{\times}) \longrightarrow {}_1\mathrm{BMod}_1(\mathcal{X}^{\times}) \simeq \mathcal{X} $$

induced by the canonical map $A \to *$. It takes an object M to $* \otimes_A M \otimes_A *$.

Remark 1.1.4. We denote $* \otimes_A M \otimes_A *$ by $A\backslash M/A$, and call it the *double quotient*. Explicitly, it is computed as the colimit over the simplicial diagram

$$ \dots \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; A^2 \times M \times A^2 \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; A \times M \times A \rightrightarrows M. $$

The monoidal structure on ${}_A\mathrm{BMod}_A(\mathcal{X}^{\times})$ is called the *twisted product* or the *diagonal action* and will be denoted by $- \times^A -$. Explicitly, $M \times^A N$ is computed as the colimit over the simplicial diagram

$$ \dots \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; M \times A^2 \times N \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; M \times A \times N \rightrightarrows M \times N $$

Lemma 1.1.5. If $\mathcal{X}$ is an ∞-topos, and $A \in \mathrm{Grp}(\mathcal{X}) \subset \mathrm{Mon}(\mathcal{X})$ is a group-object, then there is a cartesian square

$$ \begin{array}{ccc} A\backslash(M \times^A N)/A & \longrightarrow & (A\backslash M/A) \times (A\backslash N/A) \\ \downarrow & & \downarrow {\scriptstyle (\alpha_R, \alpha_L)} \\ */A & \xrightarrow[(\mathrm{id}, I)]{} & (*/A) \times (A\backslash *) \end{array} $$

for every $M, N \in {}_A\mathrm{BMod}_A(\mathcal{X}^{\times})$, where $I : */A \to A\backslash *$ denotes the equivalence provided by the inverse map $(-)^{-1} : A \xrightarrow{\sim} A$, and $\alpha_R : (A\backslash M)/A \to */A$ and $\alpha_L : A\backslash(N/A) \to A\backslash *$ are the maps classifying the right action on M and the left action on N, respectively.

Proof. This follows from [NSS14, Lemma 4.5] and the fact that the left and right actions commute, so

$$ \frac{(M/A \times A\backslash N)}{A_{\mathrm{diag}}} \simeq A\Big\backslash \frac{M \times N}{A_{\mathrm{diag}}}\Big/A \simeq A\backslash(M \times^A N)/A, $$

where A_{diag} denotes the diagonal action. □

We recall some notation from [HM24].

Notation 1.1.6. (a) Let $\Sigma^m \subset [m] \times [m]^{\mathrm{op}}$ denote the poset of pairs (i, j) such that $0 \le i \le j \le m$, and let $\mathbb{A}^m$ denote the full subposet spanned on objects (i, j) with $|i - j| \le 1$.

(b) A functor $F : \Sigma^m \to \mathcal{A}$ is *cartesian* if F is a pointwise right Kan extension of its restriction to $\mathbb{A}^m$. Equivalently, if each square is a pullback in $\mathcal{A}$. Let $\mathrm{Fun}^{\mathrm{cart}}(\Sigma^m, \mathcal{A}) \subset \mathrm{Fun}(\Sigma^m, \mathcal{A})$ denote the full subcategory spanned by cartesian functors.
Note that the restriction $\mathrm{Fun}(\Sigma^{m,n}, \mathcal{A}) \to \mathrm{Fun}(\mathbb{A}^{m,n}, \mathcal{A})$ is fully faithful ([Lur09, Theorem 4.3.2.15]), and that it induces an equivalence $\mathrm{Map}^{\mathrm{cart}}(\Sigma^{m,n}, \mathcal{A}) = \mathrm{Fun}(\Sigma^{m,n}, \mathcal{A})^{\simeq} \to \mathrm{Map}(\mathbb{A}^{m,n}, \mathcal{A})$ of anima.

(c) A functor $f : [m]^{\mathrm{op}} \times [n] \to \mathcal{A}$ is *cartesian* if it is a right Kan extension of its restriction to the poset $\mathbb{L}^{m,n} \subset [m]^{\mathrm{op}} \times [n]$ spanned on vertices (i, n) and $(0, j)$ for $i \in [m]$ and $j \in [n]$.
Let $\mathrm{Fun}^{\mathrm{cart}}([m]^{\mathrm{op}} \times [n], \mathcal{A}) \subset \mathrm{Fun}([m]^{\mathrm{op}} \times [n], \mathcal{A})$ denote the full subcategory of cartesian functors.

(d) If $\mathcal{A}$ is an ∞-category, then there is a double category $\mathrm{Sq}(\mathcal{A}) \in \mathrm{DCat}$ given by

$$ \mathrm{Sq}(\mathcal{A})([m], [n]) \simeq \mathrm{Map}([m] \times [n], \mathcal{A}) \simeq \mathrm{Fun}([m] \times [n], \mathcal{A})^{\simeq} $$

There is a double subcategory of $\mathrm{Sq}(\mathcal{A})^{1\,\mathrm{op}}$ given by

$$ \mathrm{Sq}^{\mathrm{cart}}(\mathcal{A})^{1\,\mathrm{op}}([m], [n]) \simeq \mathrm{Fun}^{\mathrm{cart}}([m]^{\mathrm{op}} \times [n], \mathcal{A})^{\simeq}. $$

Recall 1.1.7. If $(\mathcal{C}, \mathcal{C}_0)$ is a geometric setup where $\mathcal{C}$ admits finite products, then the category $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ promotes to a symmetric monoidal category

$$ \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^{\otimes} \simeq \mathrm{Corr}((\mathcal{C}^{\mathrm{op}})^{\sqcup, \mathrm{op}}, \mathcal{C}_0^{-}). $$

where $(\mathcal{C}^{\mathrm{op}})^{\sqcup,\mathrm{op}} \to \mathsf{Fin}_*$ is the coCartesian operad dual to $\mathcal{C}^\times$, and $\mathcal{C}_0^- = (\mathcal{C}_0^{\mathrm{op}})^{\sqcup,\mathrm{op}} \times_{\mathsf{Fin}_*^{\mathrm{op}}} \mathsf{Fin}_*^{\simeq}$. The symmetric monoidal structure on $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ is given by the cartesian product in $\mathcal{C}$. For example, the cocartesian lift of the map $\alpha : \langle 2 \rangle \to \langle 1 \rangle$ given by $\alpha(1) = \alpha(2) = 1$ is the span

$$\begin{array}{ccc} & X \times Y & \\ \swarrow & & \searrow^{\mathrm{id}} \\ X \oplus Y & & X \times Y \end{array}$$

1.2 Convolution

The goal is to prove the following theorem:

Theorem 1.2.1. If $\mathcal{X}$ is an ∞-topos, and $A \in \mathrm{Grp}(\mathcal{X}) \subset \mathrm{Mon}(\mathcal{X}) \simeq \mathrm{Alg}(\mathcal{X}^\times)$ is a group-object, then the oplax monoidal functor from Corollary 1.1.3

$$_A \mathrm{BMod}_A(\mathcal{X}^\times) \xrightarrow{A\backslash - /A} \mathcal{X}, \qquad M \mapsto A\backslash M/A$$

induces a lax monoidal map

$$_A \mathrm{BMod}_A(\mathcal{X}^\times)^{\otimes_A} \longrightarrow \mathrm{Corr}(\mathcal{X})^{\otimes}.$$

Consequently, there is an induced map

$$\mathrm{Mon}(\mathcal{X})_{A/} \simeq \mathrm{Alg}(\mathcal{X}^\times)_{A/} \simeq \mathrm{Alg}(_A \mathrm{BMod}_A(\mathcal{X}^\times)^{\otimes_A}) \longrightarrow \mathrm{Alg}(\mathrm{Corr}(\mathcal{X})^{\otimes})$$

so if M is a monoid-object in $\mathcal{X}$ with compatible left and right actions from A, then the double quotient $A\backslash M/A$ becomes a monoid-object in $\mathrm{Corr}(\mathcal{X})$.

Note that the equivalence $\mathrm{Alg}(\mathcal{X}^\times)_{A/} \simeq \mathrm{Alg}(_A \mathrm{BMod}_A(\mathcal{X}^\times)^{\otimes})$ says that there are no "exotic" compatible left and right actions of a monoid object A on some other monoid object M. Indeed, as these determine and are determined by a map $A \to M$.

Lemma 1.2.2. Let $F : \mathcal{O} \to \mathrm{Cat}_\infty$ be a functor with corresponding coCartesian fibration $p : \mathcal{C} \to \mathcal{O}$ and Cartesian fibration $F' : \mathcal{C}' \to \mathcal{O}^{\mathrm{op}}$.

(1) There is an orthogonal factorization system

$$\mathcal{C}_{p\text{-coCart},\mathrm{fib}} = (\mathcal{C}_{p\text{-coCart}} \to \mathcal{C} \leftarrow \mathcal{C}_{\mathrm{fib}})$$

where $\mathcal{C}_{p\text{-cocart}} \subset \mathcal{C}$ is the wide subcategory spanned on p-cocartesian lifts, and $\mathcal{C}_{\mathrm{fib}} = \mathcal{C} \times_{\mathcal{O}} \mathrm{ob}(\mathcal{O})$ is the wide subcategory consisting of only maps in the same fiber.

(2) There is an equivalence of double categories

$$\mathrm{Sq}(\mathcal{C})_{p\text{-coCart},\mathrm{fib}} \simeq \mathrm{Sq}(\mathcal{C}')^{1\,\mathrm{op}}_{q\text{-Cart},\mathrm{fib}}$$

that on $[1] \times [1]$-simplices is given by

$$\begin{array}{ccc} X & \longrightarrow & f_! X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & f_! Y \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} X & \longleftarrow & f^* X \\ \downarrow & & \downarrow \\ Y & \longleftarrow & f^* Y \end{array}$$

Proof. This is a consequence of Lurie's straightening-unstraightening. □

Lemma 1.2.3. Let $(\mathcal{A}, \mathcal{A}_0)$ be a geometric setup.

(1) The category $\mathrm{Corr}(\mathcal{A}, \mathcal{A}_0)$ promotes to an orthogonal factorization system

$$\mathrm{Corr}(\mathcal{A}, \mathcal{A}_0)_{\mathrm{Leg},\mathrm{Reg}} = (\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{Leg}} \mathrm{Corr}(\mathcal{A}, \mathcal{A}_0) \xleftarrow{\mathrm{Reg}} \mathcal{A}_0) \in \mathrm{OFS},$$

where Leg and Reg denote the left and right leg inclusions.

(2) There is an equivalence of double categories

$$\mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)_{\mathrm{Leg},\mathrm{Reg}}) \simeq \mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}$$

In particular,

$$\mathrm{Fact}(\mathrm{Corr}(\mathcal{C},\mathcal{C}_0)^{\otimes}_{\mathrm{Leg},\mathrm{Reg}}) \simeq \mathrm{Sq}^{\mathrm{cart},\mathcal{C}_0^-}((\mathcal{C}^{\mathrm{op}})^{\sqcup,\mathrm{op}}) \simeq \mathrm{Sq}^{\mathrm{cart},\mathcal{C}_0^-}(\mathcal{C}_\times)$$

Proof. (1) is evident, as any span uniquely factors as

$$\left[X \xleftarrow{f} Y \xrightarrow{g} Z \right] \simeq \left[Y \xleftarrow{\mathrm{id}} Y \xrightarrow{g} Z \right] \circ \left[X \xleftarrow{f} Y \xrightarrow{\mathrm{id}} Y \right]$$

For (2), note that $\mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([m],[n])$ sits in the cartesian square

$$\begin{array}{ccc} \mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([m],[n]) & \longrightarrow & \mathrm{Map}([m]\times[n]^{\simeq},\mathcal{A}^{\mathrm{op}}) \times \mathrm{Map}([m]^{\simeq}\times[n],\mathcal{A}_0) \\ \downarrow & \text{cart} & \downarrow{\scriptstyle (\mathrm{Leg},\mathrm{Reg})} \\ \mathrm{Map}([m]\times[n],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) & \longrightarrow & \mathrm{Map}([m]\times[n]^{\simeq},\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \times \mathrm{Map}([m]^{\simeq}\times[n],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)). \end{array}$$

There are functorial maps

$$\begin{aligned} \mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}([m],[n]) &\longrightarrow \mathrm{Map}(\mathrm{Corr}([m]^{\mathrm{op}}\times[n],[m]^{\simeq}\times[n]),\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \\ &\xrightarrow{(\mathrm{Leg}^{\mathrm{op}},\mathrm{Reg})} \mathrm{Map}([m]\times[n],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}([m],[n]) &\subset \mathrm{Map}([m]^{\mathrm{op}}\times[n],\mathcal{A}) \\ &\longrightarrow \mathrm{Map}([m]\times[n]^{\simeq},\mathcal{A}^{\mathrm{op}}) \times \mathrm{Map}([m]^{\simeq}\times[n],\mathcal{A}_0), \end{aligned}$$

which induce a map of double categories

$$\mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}} \longrightarrow \mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)$$

By the Segal condition, it suffices to show that this map is an equivalence on the bisimplices $[0]\times[0]$, $[0]\times[1]$, $[1]\times[0]$, and $[1]\times[1]$. The first three follow more or less by definition. For $[1]\times[1]$, note that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}([1],[1]) & \xrightarrow{(!)} & \mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([1],[1]) \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}(\Lambda^2_2,\mathcal{A}) & \end{array}$$

where the left map is induced by $\Lambda^2_2 = (0 \to 2 \leftarrow 1) \mapsto ((0,0) \to (0,1) \leftarrow (1,1))$ (the bottom left corner), and the right map is induced by

$$\begin{array}{ccc} \mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([1],[1]) & \longrightarrow & \mathrm{Map}([1]\times[1]^{\simeq},\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \times \mathrm{Map}([1]^{\simeq}\times[1],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \\ \vdots & & \downarrow \\ \vdots & & \mathrm{Map}([1]\times\{1\},\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \times \mathrm{Map}(\{0\}\times[1],\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\Lambda^2_2,\mathcal{A}) & \longrightarrow & \mathrm{Map}([1],\mathcal{A}) \times \mathrm{Map}([1],\mathcal{A}_0) \end{array}$$

The fiber over $(X \xrightarrow{f} Y \xleftarrow{g} Z) : \Lambda^2_2 \to \mathcal{A}$ in $\mathrm{Sq}^{\mathrm{cart},\mathcal{A}_0}(\mathcal{A})^{1\,\mathrm{op}}([1],[1])$ is contractible, as it parametrizes pullbacks of that cospan. Similarly, the fiber in $\mathrm{Fact}(\mathrm{Corr}(\mathcal{A},\mathcal{A}_0)^\dagger)([1],[1])$ is the full sub-anima on quadruples of spans satisfying

$$\left[\begin{array}{ccc} & Z & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle \mathrm{id}} \\ Y & & Z \end{array}\right] \circ \left[\begin{array}{ccc} & X & \\ {\scriptstyle \mathrm{id}'}\swarrow & & \searrow{\scriptstyle f} \\ X & & Y \end{array}\right] \simeq \left[\begin{array}{ccc} & W & \\ {\scriptstyle \mathrm{id}}\swarrow & & \searrow{\scriptstyle f'} \\ W & & Z \end{array}\right] \circ \left[\begin{array}{ccc} & W & \\ {\scriptstyle g'}\swarrow & & \searrow{\scriptstyle \mathrm{id}} \\ X & & W \end{array}\right].$$

Consequently, the square

$$\begin{array}{ccc} X & \xleftarrow{f'} & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xleftarrow[g]{} & Z \end{array}$$

is cartesian, so the fiber is contractible. Hence by the 2-out-of-3 property, the map (!) is an equivalence. □

Proposition 1.2.4. Let $p : \mathcal{C}^\otimes \to \mathsf{Assoc}^\otimes$ be a monoidal operad, and let $(\mathcal{D},\mathcal{D}_0)$ be a geometric setup such that $\mathcal{D}$ has finite limits. If $F : \mathcal{C}_\otimes \to \mathcal{D}_\times \simeq (\mathcal{D}^{\mathrm{op}})^{\sqcup,\mathrm{op}}$ is an oplax monoidal functor such that

(i) for every $f : X \simeq \bigoplus_{i=1}^m X_i \to Y \simeq \bigoplus_{i=1}^m Y_i$, $\alpha : \langle m\rangle \to \langle n\rangle$ in $\mathsf{Assoc}^\otimes$, and $i = 1,2,\dots,m$, the square

$$\begin{array}{ccc} \prod_{\alpha^{-1}(i)} F(X_i) & \longleftarrow & F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \\ \downarrow & & \downarrow \\ \prod_{\alpha^{-1}(i)} F(Y_j) & \longleftarrow & F(\bigotimes_{j\in\alpha^{-1}(i)} Y_j) \end{array}$$

is cartesian (in $\mathcal{C}$), and

(ii) $F : \mathcal{C} \to \mathcal{D}$ factors through $\mathcal{D}_0$,

then F promotes to a lax monoidal functor

$$\mathcal{C}^\otimes \longrightarrow \mathrm{Corr}(\mathcal{D},\mathcal{D}_0)^\otimes$$

that takes

(1) an object $X \simeq \oplus_{i=1}^m X_i \in \mathcal{D}^\otimes$ over $\langle m\rangle \in \mathsf{Assoc}^\otimes$ to $F(X) \simeq \oplus_{i=1}^m F(X_i) \in (\mathcal{C}^{\mathrm{op}})^{\sqcup,\mathrm{op}}$,
(2) a morphism $X \simeq \oplus_{i=1}^m X_i \xrightarrow{f} Y \simeq \oplus_{i=1}^n Y_i$ over $\alpha : \langle m\rangle \to \langle n\rangle \in \mathsf{Assoc}^\otimes$ to the span

$$\begin{array}{ccc} & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) & \\ {\scriptstyle \mathrm{oplax}}\swarrow & & \searrow{\scriptstyle F(f)} \\ \bigoplus_{i=1}^m F(X_i) & & \bigoplus_{i=1}^n F(Y_i) \end{array}$$

(3) and a 2-cell $X \xrightarrow{f} Y \xrightarrow{g} Z$ over $\langle m\rangle \xrightarrow{\alpha} \langle n\rangle \xrightarrow{\beta} \langle k\rangle$ to the span

$$\begin{array}{ccccc} & & \bigoplus_{i=1}^k F(\bigotimes_{j\in(\beta\alpha)^{-1}(i)} X_j) & & \\ & {\scriptstyle \mathrm{oplax}}\swarrow & & \searrow{\scriptstyle F(f)} & \\ & \bigoplus_{i=1}^m F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) & & \bigoplus_{i=1}^k F(\bigotimes_{j\in\beta^{-1}(i)} Y_j) & \\ {\scriptstyle \mathrm{oplax}}\swarrow & & \searrow{\scriptstyle F(f)} \quad {\scriptstyle \mathrm{oplax}}\swarrow & & \searrow{\scriptstyle F(g)} \\ \bigoplus_{i=1}^n F(X_i) & & \bigoplus_{i=1}^m F(Y_i) & & \bigoplus_{i=1}^k F(Z_i) \end{array}$$

Proof. In general, if $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ are $\mathcal{O}$-monoidal operads, and $F : \mathcal{C}_{\otimes} \to \mathcal{D}_{\otimes}$ is an oplax $\mathcal{O}$-monoidal functor (i.e., a functor of dual operads over $\mathcal{O}_{\otimes}$), then there is a map

$$\mathrm{Fact}(\mathcal{C}^{\otimes}_{p\text{-coCart,fib}}) \longrightarrow \mathrm{Sq}(\mathcal{C}^{\otimes})_{p\text{-coCart,fib}} \simeq \mathrm{Sq}(\mathcal{C}_{\otimes})^{1\,\mathrm{op}}_{q\text{-Cart,fib}} \xrightarrow{F\circ -} \mathrm{Sq}(\mathcal{D}_{\otimes})^{1\,\mathrm{op}}_{\mathrm{general,fib}}$$

where $\mathrm{Sq}(\mathcal{D}_{\otimes})^{1\,\mathrm{op}}_{\mathrm{general,fib}}$ denotes the category of squares in $\mathcal{D}_{\otimes}$ for which the vertical maps are in the same fiber.

In our setting, the two assumptions assert that this composite factors through $\mathrm{Sq}^{\mathrm{cart},\mathcal{D}_0^-}((\mathcal{D}^{\mathrm{op}})^{\sqcup,\mathrm{op}})$; indeed, we need F to take the (cartesian) square (in $\mathcal{C}_{\otimes}$) on the left to a cartesian square (in $\mathcal{D}_{\times}$) on the right:

$$\begin{array}{ccc} \bigoplus_{i=1}^m X_i & \longleftarrow & \bigoplus_{i=1}^n (\bigotimes_{j\in\alpha^{-1}(i)} X_j) \\ \downarrow & & \downarrow \\ \bigoplus_{i=1}^m Y_i & \longleftarrow & \bigoplus_{i=1}^n (\bigotimes_{j\in\alpha^{-1}(i)} Y_j) \end{array} \qquad \begin{array}{ccc} \bigoplus_{i=1}^m F(X_i) & \longleftarrow & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \\ \downarrow & & \downarrow \\ \bigoplus_{i=1}^m F(Y_i) & \longleftarrow & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} Y_j) \end{array}$$

Since $\mathcal{D}_{\times}$ is a (dual) operad, we can find cocartesian lifts factoring the square on the right as

$$\begin{array}{ccccc} \bigoplus_{i=1}^m F(X_i) & \longleftarrow & \bigoplus_{i=1}^n \prod_{j\in\alpha^{-1}(i)} F(X_j) & \longleftarrow & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} X_j) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{i=1}^m F(Y_i) & \longleftarrow & \bigoplus_{i=1}^n \prod_{j\in\alpha^{-1}(i)} F(X_j) & \longleftarrow & \bigoplus_{i=1}^n F(\bigotimes_{j\in\alpha^{-1}(i)} Y_j) \end{array}$$

The left square is cartesian, so it suffices to show that the right square is as well, but this is exactly assumption (i). So by Lemma 1.2.3.(2) and the full faithfulness of $\mathrm{Fact} : \mathrm{OFS} \to \mathrm{DCat}$ [Jur25, Theorem A], there is a map of orthogonal factorization systems

$$\mathcal{C}^{\otimes}_{p\text{-coCart,fib}} \longrightarrow \mathrm{Corr}(\mathcal{D}, \mathcal{D}_0)^{\otimes}_{\mathrm{Leg,Reg}}.$$

By construction, the underlying functor is a lax monoidal functor

$$\mathcal{C}^{\otimes} \longrightarrow \mathrm{Corr}(\mathcal{D}, \mathcal{D}_0)^{\otimes}$$

which behaves exactly as claimed in (1), (2), and (3). □

Proof of Theorem 1.2.1. It suffices to verify condition (i) of Proposition 1.2.4 as (ii) is trivially satisfied in the case at hand. For simplicity, we will treat the case $|\alpha^{-1}(i)| = 2$. From Lemma 1.1.5, the outer square and the right square are pullbacks, and therefore the left one is as well:

$$\begin{array}{ccccc} A\backslash(X_1 \times^A X_2)/A & \longrightarrow & A\backslash(Y_1 \times^A Y_2)/A & \longrightarrow & */A \\ \downarrow & & \downarrow & & \downarrow \\ A\backslash X_1/A \times A\backslash X_2/A & \longrightarrow & A\backslash Y_1/A \times A\backslash Y_2/A & \longrightarrow & */A \times A\backslash * \end{array}$$

Recall that the operadic structure on $\mathrm{LMod}_A(\mathcal{X}^{\times})^{\otimes_A}$ is the diagonal action, so this proves (i). □

Remark 1.2.5. If $D : \mathrm{Corr}(\mathcal{C})^{\otimes} \to \mathrm{Cat}^{\times}$ is a six-functor formalism (a lax symmetric monoidal map), then Theorem 1.2.1 implies that there are maps

$$\mathrm{Alg}({}_A\mathrm{BMod}_A(\mathcal{C}^{\times})^{\otimes_A}) \longrightarrow \mathrm{Alg}(\mathrm{Corr}(\mathcal{C})^{\otimes}) \longrightarrow \mathrm{Alg}(\mathrm{Cat}^{\times}) \simeq \mathrm{Mon}, \quad X \mapsto D(A\backslash X/A)^{\star}$$

We think of an object $X \in \mathrm{Alg}({}_A\mathrm{BMod}_A(\mathcal{C}^{\times})^{\otimes_A})$ as an "A-A-equivariant monoid in $\mathcal{C}$", and the monoidal category $D(A\backslash X/A)^{\star}$ as "A-A-equivariant sheaves on X". The monoidal structure $- \star -$ is called *convolution*. Explicitly, if $m : X \times^A X \to X$ is the multiplication on X, then the multiplication on $A\backslash X/A$ in

$\mathrm{Corr}(\mathcal{C})$ is given by the span

$$A\backslash X/A \times A\backslash X/A \xleftarrow{(p_1,p_2)} A\backslash(X \otimes_A X)/A \xrightarrow{m} A\backslash X/A,$$

and thus the monoidal structure on $D(X)$ is given by

$$D(A\backslash X/A) \times D(A\backslash X/A) \xrightarrow{-\boxtimes-} D(A\backslash X/A \times A\backslash X/A) \xrightarrow{(p_1,p_2)^*} D(A\backslash(X \otimes_A X)/A) \xrightarrow{m_!} D(X)$$

Example 1.2.6. Let $\mathcal{C} \simeq \mathrm{Shv}(\mathrm{Aff}_k^{\mathrm{fpqc}})$, our monoid-object is LG which is acted upon by L^+G by left and right multiplication, and D is the constructible six-functor formalism on the underlying analytic topology with a given stratification (e.g., the one in L^+G-orbits), then we get a convolution product on the double fpqc-quotient $\mathrm{Hck}_G = L^+G\backslash LG/L^+G$, or equivalently, L^+G-equivariant constructible sheaves on the affine Grassmannian $\mathrm{Gr}_G = LG/L^+G$ with L^+G-orbit stratification.

We also get a version for the Beilinson-Drinfeld affine Grassmannian.

Remark 1.2.7. The idea of embedding the problem into double categories was communicated to me by Robert Burklund.

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