Light condensed anima attached to sheaves on manifolds

This note has been written as a supplement to the fourth talk in the seminar-course TopTop[1]. Its purpose is to expand a bit on the arguments in section 5 of [Cla25].

Throughout, F is a *local field* (a non-discrete complete valued locally compact field). Recall the following classification of local fields:

Proposition 1. Let F be a local field. Then F is either

1. archimedean; in which case F is non-canonically homeomorphic to $\mathbb{R}$ or $\mathbb{C}$,
2. non-archimedean of characteristic 0; in which case F is non-canonically homeomorphic to a finite extension of $\mathbb{Q}_p$ for some prime p,
3. non-archimedean of positive characteristic; in which case F is non-canonically homeomorphic to $\mathbb{F}_{p^n}((T))$ for some prime p and natural number n.

Proof. Omitted. See remark 7.49 in [Mil20]. □

From this classification, we see that any compact subset of F^d has finite covering dimension (in the Archimedean case, it is d and $2d$, respectively, and 0 in the non-archimedean case).

Consequently, we find:

Theorem 2. If $M = (M^{top}, \mathcal{O}_M)$ is an F-manifold, then

$$\mathrm{Sh}_{\text{ét}}(\underline{M}) \xleftarrow{\sim} \mathrm{Sh}(M^{top}),$$

where $\underline{M}$ denotes the light condensed anima $\mathrm{Map}(-, M^{top}) : \mathrm{ProfSet}^{\text{light,op}} \to \mathrm{An}$.

Proof. By the preceeding text and [Lur09, Theorem 7.2.3.6], $\mathrm{Sh}(M)$ has finite homological dimension, so by [Lur09, Theorem 7.2.1.10], $\mathrm{Sh}(M)$ is postnikov-complete. We conclude by [Cla25, Lemma 4.20]. □

So to say something about étale sheaves on manifolds, we should not forget the F-analytic structure. Therefore, we make the following proposition/definition:

Proposition 3. If $\mathcal{B} \to \mathrm{Man}_F$ denote the full subcategory spanned by open polydisks, and equip Man_F with the Grothendieck topology generated by open covers, then the canonical map

$$\mathrm{Sh}(\mathcal{B}) \to \mathrm{Sh}(\mathrm{Man}_F)$$

is an equivalence. In particular, $\mathrm{Sh}(\mathrm{Man}_F)$ an ∞-topos. Furthermore, $\mathrm{Sh}(\mathrm{Man}_F)$ is hypercomplete and subcanonical.

Proof. Omitted (might add later). Uses [Hoy15, Lemma C.3]. □

Remark 4. Note that Man_F is a big category, so $\mathrm{Sh}(\mathrm{Man}_F)$ would apriori not be presentable. This is exactly the issue that the above proposition addresses.

Now we describe the "forgetful functor" from sheaves on F-manifolds to light condensed anima.

[1] https://kurser.ku.dk/course/nmak15023u

Proposition 5. 1. There is a unique colimit-preserving functor

$$\mathrm{Sh}(\mathrm{Man}_F) \to \mathrm{CondAn}^{light}$$

whose restriction to Man_F is the functor $M \mapsto \underline{M^{top}}$, where M^{top} denotes the underlying topological space. We denote this functor by $X \mapsto \underline{X}$ (abusively).

2. The functor $X \mapsto \underline{X}$ preserves pullbacks of diagrams $X \to S \leftarrow Y$ whenever $X \to S$ can, locally on S, be written as a colimit of submersions of manifolds $X_i \to S$.

Proof. For 1., we left Kan-extend along the (big) Yoneda embedding $\mathrm{Man}_F \to \mathrm{Sh}(\mathrm{Man}_F)$ to obtain the prescribed colimit-preserving functor.

For 2., we may assume that X and Y are manifolds since $\underline{(-)}$ preserves small colimits, small colimits are universal in an ∞-topos, and every (pre)sheaf is a colimit of representables. Consider the case where S is a manifold and $X \to S$ is a submersion. In this case, $X \times_S Y$ is a manifold. But as $\underline{(-)}$ restricted to Man_F preserves limits (as, in this case, $\underline{K} = \mathrm{Map}(-, K)$ for $K \in \mathrm{Man}_F$), we conclude. For general S, use the assumption that there exist a cover (U_α) of S with $U_\alpha \in \mathrm{Man}_F$ such that $X \times_S U_\alpha \to U_\alpha$ identifies with $\mathrm{colim}_I X_i \to U_\alpha$ for some $X_i \in \mathrm{Man}_F$ with $X_i \to U_\alpha$ submersions. Using universality of colimits, we reduce to the case just treated. □

Remark 6. Clausen remarks that it is reasonable to believe that $X \mapsto \underline{X}$ commutes with arbitrary pullbacks, hence giving a morphism in $\mathcal{LT}op$.

The Archimedean case

I will only consider the case when F is homeomorphic to $\mathbb{R}$. A property of real manifolds is that they are locally contractable. This gives rise to a good theory of locally constant sheaves (according to [Lur17, Appendix A]). We have a notion of "underlying anima", and this phenomenon is also present in our setting of étale sheaves.

Proposition 7. There is a unique colimit-preserving functor $|-| : \mathrm{Sh}(\mathrm{Man}_\mathbb{R}) \to \mathrm{An}$, whose restriction to $\mathcal{B}$ is $D \mapsto *$, and for any $X \in \mathrm{Sh}(\mathrm{Man}_\mathbb{R})$ there is a map $f : \underline{X} \to |X|$ in $\mathrm{CondAn}^{\mathrm{light}}$ with the following properties:

1. The map

$$\mathrm{Sh}_{\text{ét}}(|X|) \xrightarrow{f^*} \mathrm{Sh}_{\text{ét}}(\underline{X})$$

is fully faithful.

2. For any $A \in \mathrm{An}$, there is a natural equivalence

$$\mathrm{Map}(|X|, A) \xrightarrow{\sim} \mathrm{Map}(\underline{X}, A).$$

In particular, the map $f : \underline{X} \to |X|$ is natural in X.

3. For $X \in \mathrm{Sh}(\mathrm{Man}_\mathbb{R})$, the essential image $f^* : \mathrm{Sh}_{\text{ét}}(|X|) \hookrightarrow \mathrm{Sh}_{\text{ét}}(\underline{X})$ consists exactly of the locally constant sheaves on X.

4. The functor $|-|$ preserves finite products.

We are supposed to think about $|X|$ as the *underlying anima* to an $X \in \mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$. Note that 3. says that a sheaf $\mathcal{F} \in \mathrm{Sh}_{\text{ét}}(\underline{X})$ lies in the essential image if and only if there exists a cover $\{f_i : X_i \to X\}_{i \in I}$ in $\mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$ such that each $f_i^* \mathcal{F}$ is constant (pulled back along $X_i \to *$).

Proof. We define $|-|$ to be the left Kan-extension of the map $\mathcal{B} \to \mathrm{An}$, $D \mapsto *$ along the Yoneda embedding $\mathcal{B} \to \mathrm{Sh}(\mathcal{B}) \simeq \mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{D \mapsto *} & \mathrm{An} \\ {\scriptstyle y}\downarrow & \nearrow_{|-|} & \\ \mathrm{Sh}(\mathrm{Man}_{\mathbb{R}}) & & \end{array}$$

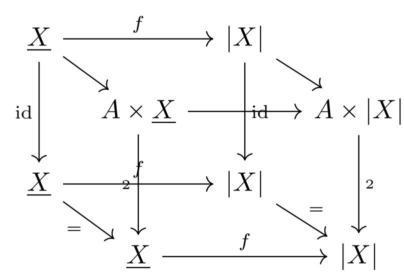
To get a map $\underline{X} \to |X|$ in $\mathrm{CondAn}^{\mathrm{light}}$, we use that $\underline{\ }$ and $|-|$ commute with small colimits to reduce to finding a map $\underline{D} \to |D| \simeq *$, which we (uniquely) have as $*$ is terminal in An.

For 1. and 3., we write $X \in \mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$ as a colimit over its disks; $X = \mathrm{colim}_{D \subset X} D$, and use that $\mathrm{Sh}_{\text{ét}}(-)$ takes colimits to limits to deduce the diagram below.

$$\begin{array}{ccc} \mathrm{Sh}_{\text{ét}}(|X|) & \xrightarrow{\simeq} & \lim_{D \subset X} \mathrm{Sh}_{\text{ét}}(|D|) \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{\text{ét}}(\underline{X}) & \xrightarrow[\simeq]{} & \lim_{D \subset X} \mathrm{Sh}_{\text{ét}}(\underline{D}) \end{array}$$

Now, the component-maps on the right identify with $\mathrm{An} \simeq \mathrm{Sh}(*) \to \mathrm{Sh}(D)$ by Theorem 2, so we conclude by Lemma 8 and Lemma 9 below.

For 2., mapping to $A \in \mathrm{An}$ corresponds to giving a section of the étale projection map $_2$: $A \times (-) \to (-)$:



so by using 1., we find

$$\mathrm{Map}(|X|, A) \simeq \mathrm{Map}_{\mathrm{Sh}_{\text{ét}}(|X|)}(|X|, A \times |X|) \simeq \mathrm{Map}_{\mathrm{Sh}_{\text{ét}}(\underline{X})}(\underline{X}, A \times \underline{X}) \simeq \mathrm{Map}(\underline{X}, A).$$

For 4., consider $X \simeq \mathrm{colim}_{D \subset X} D$ and $Y \simeq \mathrm{colim}_{D' \subset Y} D'$ in $\mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$. Using that $|-|$ preserves colimits and that colimits are universal in ∞-topoi, we reduce to $|D \times D'| \simeq |D| \times |D'|$. But this is the case, as $* \simeq * \times *$.

□

Lemma 8. If $\mathcal{C}_- : I \to \mathrm{Cat}_\infty$ be a functor, then

$$\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \lim_I \mathrm{Map}_{\mathcal{C}_i}(X_i, Y_i)$$

In particular, for a diagram of functors $(F_- : \mathcal{C}_- \to \mathcal{D}_-) : I \to \mathrm{Cat}_\infty^{\Delta^1}$ where each component $F_i : \mathcal{C}_i \to \mathcal{D}_i$ is fully faithful, the map $F = \lim_I F_i : \mathcal{C} = \lim_I \mathcal{C}_i \to \lim \mathcal{D}_i$ is fully faithful. In this situation, the essential image of F is the limit of the essential images of the F_i.

Proof. The first assertion follows since limits commute with limits and the definition of $\mathrm{Map}_{\mathcal{C}}(X, Y)$:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(X,Y) & \longrightarrow & \mathcal{C}^{\Delta^1} \\ \downarrow & \text{cart.} & \downarrow \\ * & \xrightarrow[(X,Y)]{} & \mathcal{C} \times \mathcal{C} \end{array}$$

Consequently, the horizontal maps are equivalences:

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(X,Y) & \xleftarrow{\simeq} & \lim_I \mathrm{Map}_{\mathcal{C}_i}(X_i, Y_i) \\ \downarrow & & \downarrow \simeq \\ \mathrm{Map}_{\mathcal{D}}(F(X), F(Y)) & \xleftarrow{\simeq} & \lim_I \mathrm{Map}_{\mathcal{D}_i}(F(X_i), F(Y_i)) \end{array}$$

For the last part, we note that the essential image of a fully faithful functor is equivalent to its source. ☐

Lemma 9. If D be a contractable paracompact topological space (e.g. a polydisk), then $\mathrm{An} \simeq \mathrm{Sh}(*) \xrightarrow{\pi^*} \mathrm{Sh}(D)$ is fully faithful, where $\pi : D \to *$ is the canonical map.

Proof. Fully faithfulness is equivalent to $A \to \pi_* \pi^*(A)$ being an equivalence for all $A \in \mathrm{An}$. By [Lur17, Remark A. 1.4], this identifies with the functor $\mathrm{An} \to \mathrm{An}$, $A \mapsto \mathrm{Hom}_{\mathrm{Top}}(D, |A|) \simeq \mathrm{Sing}_\bullet(\mathrm{Map}(D, |A|))$ given by

$$A \xrightarrow{\sim} \mathrm{Sing}_\bullet(|A|) \to \mathrm{Sing}_\bullet(\mathrm{Map}(D, |A|)) = \mathrm{Map}_{\mathrm{Top}}(D, |A|).$$

(here $|-|$ denotes geometric realization, and $\mathrm{Map}(D, |A|)$ is the topological space of continuous functions with compact-open topology).

But as $D \to *$ is a weak equivalence by assumption, we have a weak equivalence $|A| \simeq \mathrm{Map}(*, |A|) \xrightarrow{\sim} \mathrm{Map}(D, |A|)$, which is preserved when applying $\mathrm{Sing}_\bullet(-)$. Hence the above composition is a weak equivalence. ☐

Recall from last talk that we constructed a fully faithful functor in $\mathcal{LT}op$

$$\mathrm{An} \simeq \mathrm{Sh}(*) \xrightarrow{\delta} \mathrm{CondAn}^{\mathrm{light}},$$

whose essential image consists of those light condensed anima $X \to *$ étale over $*$. It was shown that the étale sheaves on $\delta(X)$ identify with the "local systems of anima over X".

$$\mathrm{Sh}_{\text{ét}}(\delta(X)) \simeq \mathrm{An}_{/X} \simeq \mathrm{Fun}(X, \mathrm{An}),$$

Using this, we find

Theorem 10. For $X \in \mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$, the functor

$$\mathrm{An}_{/|X|} \simeq \mathrm{Sh}_{\text{ét}}(|X|) \to \mathrm{Sh}_{\text{ét}}(\underline{X})$$

is fully faithful, and the essential image identifies with the full subcategory spanned by the locally constant sheaves on X.

Remark 11. Since both $\underline{-}$ and $|-|$ preserve finite products and colimits, they both preserve group objects and realizations.

Example 12. Let's consider $G = \mathbb{R}$. As $\mathbb{R}$ is contractable, we find

$$\mathrm{An} \simeq \mathrm{An}_{/*} \simeq \mathrm{An}_{/(*/|\mathbb{R}|)} \xrightarrow{\sim} \mathrm{Sh}_{\text{ét}}(\underline{*/\mathbb{R}}).$$

This suggests that the theory of étale sheaves on $*/\mathbb{R}$ is trivial, but it is apparently not. It will turn out that the six functor formalism we are going to construct later will do something non-trivial even in this situation. See Section 11 of [Cla25].

The non-Archimedean case (with $\mathrm{char}\, F = 0$)

In this section we are going to let F be equal to $\mathbb{Q}_p$. By the totally disconnectedness of non-Archimedean fields,

Proposition 13. If M is a compact Hausdorff $\mathbb{Q}_p$-analytic manifold, then M is isomorphic to a finite disjoint union of balls:

$$M \cong \mathbb{Z}_p^{n_1} \sqcup \cdots \sqcup \mathbb{Z}_p^{n_k}.$$

This is not that interesting, but it will be if we impose more structure; e.g. a group structure. Lazard proved in 1965 that a topological group promotes to a p-adic Lie group object if and only if it contains an open normal uniform pro-p subgroup ([Laz65]). The goal of this section is to carry this out in an in-families version. First, let me recall the classical setting. I will follow [Pst23].

Definition 14. (a) A *pro-p* group is a profinite group that can be written as a inverse limit of p-groups; i.e. $G = \varprojlim G_i$ for p-groups G_i.

(b) A pro-p group is called *powerful* if $[G, G] \leqslant \overline{G^p}$ for $p > 2$, and if $[G, G] \leqslant \overline{G^4}$ for $p = 2$.

(c) A powerful group is called *uniform* if it is torsion free.

Theorem 15 (Lazard)**.** The map

$$\mathrm{Grp}(\mathrm{Man}_{\mathbb{Q}_p}) \to \mathrm{Grp}(\mathrm{Top}), \quad G \mapsto G^{top}$$

is fully faithful, and the essential image identifies with the full subcategory spanned on those $G \in \mathrm{Grp}(\mathrm{Top})$ that admit an open subgroup $H \leqslant G$ which is uniform pro-p-group. In particular,

1. if a topological group admits a p-adic analytic structure, then it admits a unique one (as in th real case),
2. all continuous maps between p-adic analytic groups are locally analytic.

Proof. See section 19 in [Pst23] for the full proof. □

In fact, the statement about fully faithfulness holds if we replace $\mathbb{Q}_p$ by any local field F, and the underlying topological space G^{top} is also automatically Hausdorff. In the real case $F = \mathbb{R}$, we even have an equivalence

$$\mathrm{Grp}(\mathrm{Man}^{an}_{\mathbb{R}}) \to \mathrm{Grp}(\mathrm{Man}^{sm}_{\mathbb{R}}).$$

To show that every $\mathbb{Q}_p$-analytic manifold admits a subgroup that is uniform pro-p, one establishes the existence of a *standard* group. This is also the strategy for the in-families version, so I will recall the following construction:

Construction 16. (19.5 in [Pst23]) If $F(\mathbf{X}, \mathbf{Y})$ is an n-dimensional formal group law over the p-adic integers; that is,

$$F_i(\mathbf{X}, \mathbf{Y}) = \sum_{I,J} a_{i,I,J}\mathbf{X}^I\mathbf{Y}^J \in \mathbb{Z}_p[\![\mathbf{X}, \mathbf{Y}]\!],$$

where I, J are multi-indices, then for $\mathbf{x}, \mathbf{y} \in p \cdot \mathbb{Z}_p^{\times n}$, the series

$$F_i(\mathbf{x}, \mathbf{y}) = \sum_{I,J} a_{i,I,J}\mathbf{x}^I\mathbf{y}^J$$

converges for each $1 \leqslant i \leqslant n$. Since F is a formal group law, this produces an associative multiplication

$$S \times S \to S$$

with $0 = (0, \ldots, 0)$ as a two-sided unit, where $S = p \cdot \mathbb{Z}_p^n$. One may show that this multiplication makes S into a group. The group law is, by construicition, p-adic analytic, and S promotes to a p-adic analytic manifold. We call S the *standard group associated to* F if $p > 2$, and for $p = 2$, one should consider $S = 4 \cdot \mathbb{Z}_2^n$.

For $p = 2$, we choose to consider $4 \cdot \mathbb{Z}_2^n$ instead of $2 \cdot \mathbb{Z}_2^n$ because we want the following proposition to hold:

Proposition 17. Every standard group is uniform pro-p.

Finally, we recall a few properties of uniform pro-p groups that will be used later (in the proof of Proposition 21):

Proposition 18. For a uniform pro-p group, we have the following properties, although we are only going to use 1. and 2. in this section of the paper.

1. The subset of p^n-powers $H^{p^n} \subset H$ is an open normal subgroup, and $\bigcap H^{p^n} = \{e\}$.
2. Each $H^{p^n}/H^{p^{n+1}}$ is abelian, hence an $\mathbb{F}_p$-vector space, and has dimension d where $d = \dim(H) = \dim(G)$;

Proof. Omitted. □

Now we begin to work towards an in-families version. For that, we need the following technical lemma:

Lemma 19. Let $m, n \geqslant 0$ and let $\phi : U \to \mathbb{Q}_p^m$ be a $\mathbb{Q}_p$-analytic map from an open subset $U \subset \mathbb{Q}_p^n$ containing 0 such that $\phi(0) = 0$ and the $d\phi_0(\mathbb{Z}_p^n) \subset \mathbb{Z}_p^m$ where $d\phi_0 : \mathbb{Q}_p^n \to \mathbb{Q}_p^m$ is map on tangent spaces. There exists an $N \geqslant 0$ such that

1. $p^N\mathbb{Z}_p^n \subset U$,

2. power-series for ϕ at 0 converges on $p^N\mathbb{Z}_p^n$,

3. the coordinate-change $X' = p^N X$ for both source and target ensures the coefficients of the Taylor-expansion for ϕ (in the new coordinates) are in $\mathbb{Z}_p$.

4. $\phi(\mathbb{Z}_p^n) \subset \mathbb{Z}_p^m$.

Proof. Working coordinate-wise, assume $m = 1$. We may pick N big so the power-series for ϕ converges in the ball $p^N\mathbb{Z}_n^p = \{X \| X_i| \leqslant p^{-N}\}$:

$$\phi(X) = \sum_I c_I X^I \quad \text{and} \quad \sum_I |c_I| p^{-N(i_1+\cdots+i_n)} < \infty$$

(for $I = (i_1, \cdots, i_n)$, where $i_k \in \mathbb{N}\}$). In particular, for all but finitely many I, $c_I p^{N(i_1+\cdots+i_n)} \in \mathbb{Z}_p$. As $\phi(0) = 0$, we may pick N big enough to ensure this for all I.

Coordinate-change $X' = p^N X$ does the following to the c_I:

$$c'_I = \frac{1}{p^M} c_I p^{M(i_1+\cdots+i_n)}$$

By assumption, the linear terms are in $\mathbb{Z}_p$. For the other terms, $i_1 + \cdots + i_n \geqslant 2$, so take $M = 2N$, we get all $c'_I \in \mathbb{Z}_p$, whence 1., 2., 3. Finally, 4. follows from $\mathbb{Z}_p$ being closed under convergent sums in $\mathbb{Q}_p$.

□

We get an in-families version of the uniform pro p-subgroup object:

Proposition 20. If M be a compact Hausdorff p-adic manifold, and G be group object in $(\mathrm{Man}_{\mathbb{Q}_p})_{/M}$ such that the map $G \to M$ is a submersion, then exists a compact Hausdorff subgroup object $H \subset G$ over M such that for all $x \in M$, the fiber H_x is a uniform pro-p group.

Proof. By Proposition 13 we are free to work locally, as gluing up will be tautological; so assume $M = \mathbb{Z}_p^e$. Implicit function theorem tells us that we can write the submersion $G \to M$ in local coordinates around the identity section as the projection off of M,

$$\mathbb{Z}_p^d \times \mathbb{Z}_p^e \to \mathbb{Z}_p^e,$$

such that identity section corresponds to $t \mapsto (0, t)$. Consider $\phi : G \times_M G \to G$, $(g, h) \mapsto gh^{-1}$. An open subset $H \subset G$ containing the identity section is a subgroup object if and only if $\phi(H \times_M H) \subset H$. On the chart, ϕ takes on the form

$$(\mathbb{Z}_p^d \times \mathbb{Z}_p^e) \times_{\mathbb{Z}_p^e} (\mathbb{Z}_p^d \times \mathbb{Z}_p^e) \simeq \mathbb{Z}_p^d \times \mathbb{Z}_p^d \times \mathbb{Z}_p^e \to G$$

over $M = \mathbb{Z}_p^e$. By continuity, there is $k \geqslant 0$ such that $\phi(p^k(\mathbb{Z}_p^d \times \mathbb{Z}_p^d \times \mathbb{Z}_p^e)) \subset \mathbb{Z}_p^d \times \mathbb{Z}_p^e$. As $\phi(0) = 0$ and $d\phi_0(X, Y, Z) = X - Y + Z$, the previous lemma implies existence of a new (smaller) chart where we have

$$\phi(\mathbb{Z}_p^d \times \mathbb{Z}_p^d \times \mathbb{Z}_p^e) \subset \mathbb{Z}_p^d \times \mathbb{Z}_p^e$$

and the power-series ϕ has $\mathbb{Z}_p$-coefficients.

Hence $\mathbb{Z}_p^d \times \mathbb{Z}_p^e$ is a compact open subgroup object of G. The fiber over $x \in \mathbb{Z}_p^e$ is a group with group law defined by a convergent formal group law with $\mathbb{Z}_p$-coefficients, so if we further restrict to $(2p\mathbb{Z}_p^d) \times \mathbb{Z}_p^e$, we get fibers that are standard groups, which are uniform pro-p groups by Proposition 17. □

The following result is the main takeaway form this section:

Proposition 21. Let M be a compact Hausdorff $\mathbb{Q}_p$-manifold, let $G \to M$ be a group object in submersions over M, and let $H \subseteq G$ be a compact Hausdorff open subgroup object such that each fiber H_x is a uniform pro-p group (whose existence is guaranteed by the previous result). Let $BH \in \mathrm{Sh}(\mathrm{Man}_{\mathbb{Q}_p})_{/M}$ be the classifying stack of H relative to M, and similarly for H^{p^n} or G. Then:

1. For $n \geqslant 1$, the subset $H^{p^n} \subseteq G$ is also a compact open sub-group object.
2. $\underline{BH} \to \underline{BG}$ and $\underline{BH^{p^n}} \to \underline{BH}$ for all $n \geqslant 0$ are étale.
3. $\underline{BH} = \lim_{\mathbb{N}^{\mathrm{op}}} \underline{B(H/H^{p^n})}$.
4. $\underline{BH}$ is a 1-truncated light profinite anima.

Proof. For 1., H^{p^n} is by definition the image of continuous map $H \to H$, $g \mapsto g^{p^n}$, so compact as H is compact. It is a subgroup object as it is so fiberwise (property of uniform pro-p groups). H Hausdorff so H^{p^n} is closed. Hence H/H^{p^n} is the quotient of a compact Hausdorff space by a closed equivalence relation, hence compact Hausdorff. We claim that $H/H^{p^n} \to M$ is finite étale (a finite covering map). Indeed, it is a map of compact Hausdorff spaces, and it has finite fibers whose cardinality is a locally constant function of the base:

$$|H_x/H_x^{p^n}| = |H_x/H_x^p||H_x^p/H_x^{p^2}|\dots|H_x^{p^{n-1}}/H_x^{p^n}| = p^{dn},$$

where $d = \dim(H_x) = \dim(G_x)$. Whence $H^{p^n} \subset H$ is open.

For 2., we claim that if $G' \subset G$ is an open subgroup object, then $\underline{BG'} \to \underline{BG}$ is étale. First, note that $G/G' \to M$ is a local homeomorphism. Indeed, as $G \to M$ is a submersion, it admits local sections (implicit function theorem), and hence $G/G' \to M$ does so as well. We have the diagram on the left which induces the diagram on the right by Proposition 5 and [Cla25, Lemma 4.20]:

$$\begin{array}{ccc} G/G' & \longrightarrow & BG' \\ {\scriptstyle\text{(finite) local homeo.}}\downarrow & \text{cart} & \downarrow \\ M \simeq G/G & \twoheadrightarrow & BG \end{array} \qquad \begin{array}{ccc} \underline{G/G'} & \longrightarrow & \underline{BG'} \\ {\scriptstyle\text{étale}}\downarrow & \text{cart} & \downarrow \\ \underline{M} & \twoheadrightarrow & \underline{BG} \end{array}$$

We conclude as étaleness can be checked along an effective epimorphism (see [Cla25, Lemma 4.17]).

For 3., note that we have a cartesian square

$$\begin{array}{ccc} BH^{p^n} & \longrightarrow & M \\ \downarrow & \text{cart} & \downarrow\!\!\downarrow \\ BH & \longrightarrow & B(H/H^{p^n}) \end{array}$$

which is again cartesian when applying $\underline{\ }$. Consequently, as limits commute with limits,

$$\begin{array}{ccc} \lim_{\mathbb{N}^{\mathrm{op}}} \underline{BH^{p^n}} & \longrightarrow & \underline{M} \\ \downarrow & \text{cart} & \downarrow\!\!\downarrow \\ \underline{BH^{p^n}} & \longrightarrow & \lim_{\mathbb{N}^{\mathrm{op}}} \underline{B(H/H^{p^n})} \end{array}$$

is cartesian. Here we used repleteness so see that $\underline{M} \to \lim_{\mathbb{N}^{op}} \underline{B(H/H^{p^n})}$ is an effective epimorphism. So it suffices to show that $\lim_{\mathbb{N}^{op}} \underline{BH^{p^n}} \to \underline{M}$ is an equivalence. We consider the section $\underline{M} \to \lim_{\mathbb{N}^{op}} \underline{BH^{p^n}}$ induced by the compatible sections $\underline{M} \to \underline{BH^{p^n}}$ (that picks out the identity). Again, by repleteness (and limits commute with limits), the diagram on the left gives the one on the right, where the lower horizontal map is an effective epimorphism:

$$\begin{array}{ccc} H^{p^n} & \longrightarrow & M \\ \downarrow & \text{cart} & \downarrow \\ M & \twoheadrightarrow & BH^{p^n} \end{array} \qquad \begin{array}{ccc} \lim_{\mathbb{N}^{op}} \underline{H^{p^n}} & \longrightarrow & \underline{M} \\ \downarrow & \text{cart} & \downarrow \\ \underline{M} & \twoheadrightarrow & \lim_{\mathbb{N}^{op}} \underline{BH^{p^n}} \end{array}$$

So it suffices to show that $\lim_{\mathbb{N}^{op}} H^{p^n} \to M$ is an equivalence, which we will do fiberwise. In that case, it follows form assumption on H and Proposition 18:

$$\begin{array}{ccc} \{e\} = \bigcap H_x^{p^n} = \lim_{\mathbb{N}^{op}} H_x^{p^n} & \longrightarrow & \lim_{\mathbb{N}^{op}} H^{p^n} \\ \simeq\downarrow & \text{cart} & \downarrow \\ * & \xrightarrow[x]{} & M \end{array}$$

For 4., it suffices to show that each $\underline{B(H/H^{p^n})}$ is a 1-truncated light profinite anima (as both $\mathrm{ProfAn}^{\mathrm{light}}$ and $\mathrm{ProfAn}^{\mathrm{light}}_{\leqslant 1}$ are closed under countable limits). To do so, we will show that $\underline{B(H/H^{p^n})} \to \underline{M}$ is π-finite étale, and then conclude by Lemma 4.26. But we showed that $H/H^{p^n} \to M$ has finite fibers of locally constant cardinality, hence locally on M, $\underline{B(H/H^{p^n})} \to \underline{M}$ identifies with $\underline{M} \times \underline{B(H_x/H_x^{p^n})} \to \underline{M}$ which is the projection off some π-finite anima $\underline{B(H_x/H_x^{p^n})}$, ☐

Thus, for any compact Hausdorff $M \in \mathrm{Man}_{\mathbb{Q}_p}$ and any Lie group object $G \to M$ in submersion over M, the relative classifying stack BG has the following property: étale locally on source, the associated light condensed anima is light profinite. More generally one can deduce that the same claim holds for any quotient stack or relative quotient stack. Clausen conjectures that this actually holds for all *p-adic analytic smooth Artin stacks*:

Definition 22. An F-analytic stack $X \in \mathrm{Sh}(\mathrm{Man}_F)$ is a *smooth Artin stack* if there is an $M \in \mathrm{Man}_F$ and a representable surjective submersion $M \to X$.

Remark 23. Recall that a representable map $Y \to X$ has property (P) if it does so on pullback along test objects. Clausen writes “surjective representable submersion”, where the property “surjective” is not to be understood as a property (P) in the sense above, but in the ∞-topos sense. It turns out that these notions agree when the representable map is a submersion.

Conjecture 24. If X be a p-adic analytic smooth Artin stack, then étale locally, the associated light condensed anima is a light profinite anima.

Example 25. 1. For $G \in \mathrm{Grp}(\mathrm{Man}_F)$, the classifying stack $BG \simeq */G$ is a F-analytic smooth artin stack. Indeed, consider below diagram with $X \in \mathrm{Man}_F$. $P \to X$ is a principal G bundle over X, and is thus locally trivial. As X is a manifold, we may pick a cover with $U_i \in \mathrm{Man}_F$ that pulled back to P trivializes; i.e. the manifolds $U_i \times G$ cover X, and hence X is a manifold. The map $P \to X$ is an effective epimorphism of manifolds, which is equivalent to submersion.

It is surjective (as sets) as it is so locally.

$$\begin{array}{ccccc} U_i \times G & \longrightarrow & P & \longrightarrow & * \\ \downarrow & & \twoheaddownarrow & & \twoheaddownarrow \\ U_i & \longrightarrow & X & \longrightarrow & [*/G] \end{array}$$

2. More generally, if $N \in \mathrm{Man}_F \subset \mathrm{Sh}(\mathrm{Man}_F)$ comes with a G-action, then $N \to N/G$ exhibits N/G as a F-analytic smooth Artin stack. The proof reduces to above.

3. Even more generally, if $M \in \mathrm{Man}_F$ and $G \in \mathrm{Man}_F$ is as group object with a submersion $G \to M$, then for any manifold. N over M, $N \to M$, with a compatible action of G, we have N/G is an F-analytic smooth Artin stack, called the relative quotient stack (here N/G is formed in the slice-topos).

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