The solid six-functor formalism

This note has been written as a supplement to the thirteenth and fourteenth talk in the seminar-course TopTop[1]. Its purpose is to explain the solid six-functor formalism on schemes of finite type of finite type over $\mathbb{Z}$. It is loosely based on (some handwritten notes taken during) a talk given by Dustin Clausen in Luminy.

Notation 1. By *of finite type*, I mean *of finite type over* $\mathbb{Z}$.

Notation 2. By $D(\mathbb{Z})$ I mean $D(\mathrm{Cond}(\mathrm{Ab})^{\mathrm{light}})$, where Ab is the category of (classical) abelian groups, and for a animated light condensed ring R, we write $D(R)$ for $\mathrm{Mod}_{R^\circ}(D(\mathbb{Z}))$, where $(-)^\circ$ is the map

$$\mathrm{An}(\mathrm{CAlg}(\mathrm{Cond}(\mathrm{Ab})^{\mathrm{light}})) \longrightarrow \mathrm{CAlg}(\mathrm{An}(\mathrm{Cond}(\mathrm{Ab})^{\mathrm{light}})) \simeq \mathrm{CAlg}(D_{\geqslant 0}(\mathbb{Z})).$$

Introduction

To produce a six-functor formalism on the category $\mathrm{Sch}^{\mathrm{ft}} = \mathrm{Sch}^{\mathrm{ft}}_{\mathbb{Z}}$ of schemes of finite type, we use the machinery of Liu-Zheng nicely formalised by Heyer-Mann in [HM24]. We aim to show the following theorem:

Theorem 3. There is a a six-functor formalism

$$D_{\square} : \mathrm{Corr}(\mathrm{Sch}^{\mathrm{ft}}, E) \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{\mathrm{st}})$$

where $E = \{\text{separated and of finite type}\}$, such that

- (i) In pullback functoriality, $D_{\square} : \mathrm{Sch}^{\mathrm{ft,op}} \to \mathrm{CAlg}(\mathrm{Pr}^{L}_{\mathrm{st}})$ satisfies Zariski descent,
- (ii) If $X \simeq \operatorname{Spec} A$ is affine, then $D_{\square}(X) \subset D(A) \simeq \mathrm{Cond}(D(\mathrm{Mod}^{\heartsuit}_{A}))^{\mathrm{light}}$ is the reflexive subcategory generated under colimits by objects on the form $A^I \simeq \prod_{i\in I} A$.
- (iii) If $f : Y \to X \in \mathrm{Sch}^{\mathrm{ft}}$ is proper, then there is a canonical equivalence $f_! \xrightarrow{\sim} f_*$.
- (iv) If f is of finite Tor-dimension, then f is suave. In particular, $f^!$ satisfies base-change, and $f^!(1) \otimes f^*(-) \xrightarrow{\sim} f^!(-)$. Also, if $f : Y \to X$ is furthermore smooth of relative dimension d, then $f^!(1) \simeq \Omega^d_{Y/X}[d]$.
- (v) There are full subcategories
$$D_{\delta}(X) \subset D_{\square}(X) \supset D_{\mathrm{pc}}(X)$$
of *discrete* and respectively *pseudo-coherent* modules, both of which are subsheaves wrt. the pullback functoriality, and if $X \simeq \mathrm{Spec}(A)$ is affine, then
$$D_{\delta}(X) \simeq D(\mathrm{Mod}^{\heartsuit}_{A}) \simeq \langle A \rangle \subset D_{\square}(A)$$
is the full stable ∞-category generated by A under colimits, and
$$D_{\mathrm{pc}}(X) \subset D_{\square}(A)$$
is the full subcategory spanned by those M such that
$$\bigoplus_{i\in I} \underline{\mathrm{Hom}}(M, N_i) \to \underline{\mathrm{Hom}}(M, \bigoplus_{i\in I} N_i)$$
is an equivalence for all $(N_i)_{i\in I}$ in $D_{\square,\leqslant n}(A)$; that is, tuples $(N_i)_{i\in I}$ for which there exists some n such that $\pi_*(N_i) = 0$ for $* > n$ for all $i \in I$.

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[1] https://kurser.ku.dk/course/nmak15023u

(vi) For any $f : Y \to X$ in $\mathrm{Sch}^{\mathrm{ft}}$, we have

$$f_*(D_\delta(Y)) \subset D_\delta(X) \quad \text{and} \quad f_!(D_{\mathrm{pc}}(Y)) \subset D_{\mathrm{pc}}(X).$$

Remark 4. Mann has told Hesselholt who has told me that the converse of Theorem 3.(iv) holds: $f : Y \to X$ is suave if and only if f is of finite Tor-dimension.

Also, it should be possible to slightly massage the arguments to get rid of the "of finite type" hypothesis.

Let's investigate some immediate consequences.

Definition 5. Let X be a scheme. Define $D^-_{\mathrm{Coh}}(X) = D_\delta(X) \cap D_{\mathrm{pc}}(X) \subset D_{\square}(X)$, and let $\mathrm{Perf}(X) \subset D_{\square}(X)$ be the full subcategory spanned by the perfect objects, or equivalently, the compact objects.

Remark 6. We should think about this as "finite" is equal to "discrete" and "compact".

Corollary 7. (i) Finiteness of proper pushforward: If $f : Y \to X$ is proper, then

$$f_*(D^-_{\mathrm{Coh}}(Y)) \subset D^-_{\mathrm{Coh}}(X).$$

(ii) Serre duality: If $Y \to X$ is smooth and proper, then

$$f_*(\mathrm{Perf}(Y)) \subset \mathrm{Perf}(X),$$

and for $P \in \mathrm{Perf}(Y)$, there is a canonical equivalence

$$f_*(P)^\vee \simeq f_*(P^\vee \otimes \Omega^d_{Y/X}[d]).$$

More generally, if $f : Y \to X$ is smooth but not necessarily proper, then for all $P \in \mathrm{Perf}(Y)$, $f_!(P) \in \mathrm{Perf}(X)$, and there is a canonical equivalence

$$f_!(P)^\vee \simeq f_*(P^\vee \otimes \Omega^d_{Y/X}[d]).$$

Proof. For (i), we combine Theorem 3.(iii) and Theorem 3.(vi).

For (ii), we use linearity to identify

$$\mathrm{Hom}(f_!(P), -) \simeq \mathrm{Hom}(P, f^!(-)) \simeq \mathrm{Hom}(P, f^!(1) \otimes f^*(-)),$$

so $f_!$ preserves compact objects, as as both f^* and $- \otimes -$ preserves colimits.

From [HM24, Lemma B.1.16], we see that if $P \in D_{\square}(Y)$ is dualizable then $P^\vee \simeq \underline{\mathrm{Hom}}(P, 1)$, so using smoothness, we can identify

$$f_!(P)^\vee \simeq \underline{\mathrm{Hom}}(f_!(P), 1) \simeq f_*\underline{\mathrm{Hom}}(P, f^!(1)) \simeq f_*(P^\vee \otimes f^!(1)) \simeq f_*(P^\vee \otimes \Omega^d_{Y/X}[d]).$$

If $f : Y \to X$ is also proper, then $f_*(P)^\vee \simeq f_!(P)^\vee \simeq f_*(P^\vee \otimes \Omega^d_{Y/X}[d])$. □

The idea to prove Theorem 3 is to embed schemes of finite type over $\mathbb{Z}$ into discrete Huber pairs (of finite type) and construct the six-functor formalism there. More precisely, we will

- construct fully faithful functors

$$\begin{tikzcd}
& \mathrm{Huber}^{\mathrm{ft}} \arrow[dr, hook, "{(A,A^+)\mapsto(A,A^+)_\square=(A,D_\square(A,A^+))}"] & \\
\mathrm{Ring}^{\mathrm{ft}} \arrow[ur, hook, "{A\mapsto(A,A)}"] \arrow[rr, hook, "{A\mapsto A_\square=(A,D_\square(A))}"'] & & \mathrm{AnRing}
\end{tikzcd}$$

  and use the forgetful functor

$$\mathrm{AnRing} \xrightarrow{A=(A^{\triangleright},D(A))\mapsto D(A)} \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$$

  to construct the pullback-functoriality

$$D_\square(-) : \mathrm{AffAdicSpaces}^{\mathrm{ft,op}} \simeq \mathrm{Huber}^{\mathrm{ft}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}),$$

  which we may restrict to

$$D_\square : \mathrm{Aff}^{\mathrm{ft,op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}).$$

- Show that $D_\square$ promotes to a six-functor formalism on $\mathrm{AffAdicSpaces}^{\mathrm{ft}} \simeq \mathrm{Huber}^{\mathrm{ft,op}}$ using [HM24, Proposition 3.3.3].
- Show that $D_\square$ extends to a six-functor formalism on $\mathrm{Sh}(\mathrm{AffAdicSpaces}^{\mathrm{ft}})$ using [HM24, Theorem 3.4.11].
- Restrict this six-functor formalism to the full subcategory of schemes of finite type over $\mathbb{Z}$, $\mathrm{Sch}^{\mathrm{ft}} \subset \mathrm{Sh}(\mathrm{Aff}^{\mathrm{ft}}) \subset \mathrm{Sh}(\mathrm{AffAdicSpaces}^{\mathrm{ft}})$, by exhibiting a map of corresponding geometric setups.

Rings and discrete Huber pairs

In this subsection, we describe discrete Huber pairs and their embedding into analytic rings. First, we will describe how to give a discrete animated condensed ring of finite type the structure of an analytic ring.

Theorem 8 (Scholze)**.** If A is a discrete animated ring of (almost) finite type, then the subcategory $D_{\geqslant 0}(A_\square) \subset D_{\geqslant 0}(A)$ generated under colimits by $\prod_I A$ for varying $\aleph_1$ sets I defines an analytic ring structure on A, denoted $A_\square = (A, D_{\geqslant 0}(A_\square))$.

The completion of $A[S] \in D_{\geqslant 0}(A)$ for $S = \lim_i S_i$ is given by the static module $\lim_i A[S_i] \in D_{\geqslant 0}(A)$, and this object is equivalent to $\prod_I A$ for some set I.

The objects $\prod_I A$ are compact projective generators of $D_{\geqslant 0}(A_\square)$, and is stable under tensor-product in the following sense:

$$\prod_I A \otimes \prod_J A \simeq \prod_{I\times J} A.$$

Proof. See [Sch25, Theorem 9.6]. □

Remark 9. We will write $D_\square(A)$ for the stabilization of $D_{\geqslant 0}(A_\square)$ and refer to its objects as *solid quasicoherent A-modules*. Note that $D_\square(A) \simeq D(A_\square)$ is compactly generated, and the subcategory of compact objects $D_\square(A)^\omega$ is generated under finite colimits, shifts and retracts by A^I for varying sets I.

Definition 10. (a) Let $\mathrm{Ring}^{\mathrm{ft}}$ be the full subcategory of animated condensed rings spanned by (discrete animated condensed) rings of (almost) finite type.

(b) A *discrete Huber pair* is a pair (A, A^+) where A is a discrete animated ring and A^+ is an integrally closed subring of $\pi_0 A$. A morphism $f : (A, A^+) \to (B, B^+)$ of discrete Huber pairs is a morphism $f : A \to B$ such that $\pi_0 f : A^+ \to \pi_0 B$ factors through B^+. A discrete Huber pair is said to be of *of finite type* if A^+ is of finite type (over $\mathbb{Z}$). These form a category $\mathrm{Huber}^{\mathrm{ft}}$.

Construction 11. To a Huber pair of finite type (A, A^+) where A is a static ring (i.e. a usual ring), we associate an analytic ring $(A, A^+)_\square$ with underlying ring A by letting $D((A, A^+)_\square)$ be the full subcategory of $D(A)$ on objects M that factor through $D_\square(A^+)$ under the restriction map $D(A) \to D(A^+)$ induced by $A^+ \subset \pi_0 A \to \pi_0 A \simeq A$. If A is not ordinary, then we use **??** to induce a structure from $(\pi_0 A, A^+)_\square$. In other words, $D((A, A^+)_\square) = \mathrm{Mod}_A(D_\square(A^+))$.

Remark 12. It turns out that one need not require A^+ to be integrally closed in $\pi_0 A$ to make the construction above, but the analytic ring turns out to coincide with $(A, \widetilde{A^+})_\square$, where $\widetilde{A^+}$ denotes the integral closure of A^+ in $\pi_0 A$. We stick with the integral closure in the definition as we want Theorem 13 to hold.

If A is an ordinary ring, then the completion functor is given by

$$(A, A^+)_\square[S] \simeq A^+_\square[S] \otimes_{A^+} A.$$

for profinite sets S.

Theorem 13. There are fully faithful functors

$$\begin{tikzcd}
& \mathrm{Huber}^{\mathrm{ft}} \arrow[dr, hook, "{(A,A^+)\mapsto(A,A^+)_\square=(A,D_\square(A,A^+))}"] & \\
\mathrm{Ring}^{\mathrm{ft}} \arrow[ur, hook, "{A\mapsto(A,A)}"] \arrow[rr, hook, "{A\mapsto A_\square=(A,D_\square(A))}"'] & & \mathrm{AnRing},
\end{tikzcd}$$

each preservering non-empty colimits.

Proof. The functor $\mathrm{Ring}^{\mathrm{ft}} \to \mathrm{Huber}^{\mathrm{ft}}$ is easily seen to be fully faithful and colimit preserving. For the functor $\mathrm{Huber}^{\mathrm{ft}} \to \mathrm{AnRing}$, see [Sch25, Theorem 9.8] and [Man22, Proposition 2.9.6]. □

Construction of the solid six-functor formalism

In this section we will upgrade the functor $D_\square(-) : \mathrm{Huber}^{\mathrm{ft}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ to a six-functor formalism where every map is !-able. Recall the general procedure of [HM24, Proposition 3.3.3]: We must

(1) provide a suitable decomposition (I, P) of $E = \{\mathrm{all}\}$,

(2) show that for every $f \in P$, f^* admits a right adjoint f_* that satisfies base-change and the projection formula,

(3) show that for every $j \in I$, j^* admits a left adjoint $j_!$ that satisfies base-change and the projection formula, and

(4) justify a technical equivalence on how maps from I and P interact.

Remark 14. In view of [HM24, Corollary 3.3.5 and Remark 3.3.6] and the suitable decomposition below, the (d) is redundant.

To get started, note that every map $(A, A^+) \to (B, B^+)$ of discrete Huber pairs factors canonically as

$$\begin{tikzcd}
(A, A^+) \arrow[rr, "f"] \arrow[dr] & & (B, B^+) \\
& (B, A^+) \simeq (B, \widetilde{\operatorname{im} \pi_0 f}) \arrow[ur] &
\end{tikzcd}$$

which suggests the following definition:

Definition 15. A map of discrete Huber pairs $f : (R, R^+) \to (S, S^+)$ is

(a) an *open immersion* if $R \to S$ is an equivalence, and S^+ is generated over R^+ by finitely many elements, and

(b) a *proper map* if S^+ is the integral closure of the image of $\pi_0 f$.

Let I be the class of open immersions and P the class of proper maps.

This takes care of (a):

Proposition 16. (I, P) is a suitable decomposition of $E = \{\text{all}\}$, and all open immersions are monomorphisms.

Proof. Both I and P evidently contain all identity morphisms and are stable under composition and pullback. The paragraph above shows that every map in E admits a factorization of a morphism from I and P. It is also easy to check left-cancelable. ☐

For (b), we already know that f^* admits a right adjoint f_* which satisfy the projection-formula (see **??**). Therefore, it suffices to establish base-change:

Proposition 17. If $f : (A, A^+) \to (B, A^+)$ be a proper map, then the f is steady (i.e. f_* satisfies base-change).

Proof. Note that B is discrete as an object of $D_{\square}(A, A^+)$ (i.e. it can be written as a colimit of copies of A; specifically, we have a sequence $\bigoplus_B A \to B \to 0$). The condition in [Sch20, Proposition 13.9] preserves colimits, and is tautologically true for $A \simeq B$, so it follows that f is steady. ☐

For (c), the crucial step is the construction of $j_!$ where j is the map $(\mathbb{Z}[T], \mathbb{Z}) \to (\mathbb{Z}[T], \mathbb{Z}[T])$. The following to lemmas are the technical machine behind the construction:

Lemma 18. Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category that is closed. If $F \to 1 \to A$ is a fiber sequence in $\mathcal{C}$ with A an idempotent object of $\mathrm{CAlg}(\mathcal{C})$, then there is a stable recollement

$$\begin{tikzcd}
\mathrm{Mod}_A(\mathcal{C}) \arrow[r, "i_* \simeq i_!"] & \mathcal{C} \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[r, "j^! \simeq j^*"] & \mathrm{coMod}_F(\mathcal{C}) \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"]
\end{tikzcd}$$

and we have the following formulas

$$i_* i^*(-) \simeq - \otimes A, \qquad j_! j^*(-) \simeq - \otimes F,$$
$$i_* i^!(-) \simeq \underline{\mathrm{Hom}}(A, -), \qquad j_* j^*(-) \simeq \underline{\mathrm{Hom}}(F, -),$$

and the following sequences

$$j_! j^* X \to X \to i_* i^* X \quad \text{and} \quad i_* i^! X \to X \to j_* j^* X \tag{1}$$

for any $X \in \mathcal{C}$.

Proof. See [CS22, Chapter 5] or [CDH$^+$25, Appendix A]. $\square$

Lemma 19. If $\mathcal{C} \to \mathcal{D}$ be a fully faithful functor in $\mathrm{Pr}^L_{\mathrm{st}}$, then the class S is morphisms $f : X \to Y$ in $\mathcal{D}$ whose fiber lies in (the essential image of) $\mathcal{C}$ is a strongly saturated class and the Verdier quotient $\mathcal{D}/\mathcal{C}$ is equivalent to the Bousfield localization $S^{-1}\mathcal{D}$.

Proof. See [BGT13, Proposition 5.6]. $\square$

Theorem 20. If j denotes the map $(\mathbb{Z}[T], \mathbb{Z}) \to (\mathbb{Z}[T], \mathbb{Z}[T])$, then j^* admits a left adjoint $j_!$. More precisely, there is a stable recollement

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{Z}((T^{-1}))}(D_{\square}(\mathbb{Z}[T], \mathbb{Z})) \arrow[r, "i_* \simeq i_!"] & D_{\square}(\mathbb{Z}[T], \mathbb{Z}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D_{\square}(\mathbb{Z}[T], \mathbb{Z}[T]). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

Furthermore,

(i) j is steady:

$$\begin{tikzcd}
(B, B^+) & (\mathbb{Z}[T], \mathbb{Z}[T]) \arrow[l, "g'"'] \\
(B, A^+) \arrow[u, "j'"] & (\mathbb{Z}[T], \mathbb{Z}) \arrow[l, "g"] \arrow[u, "j"]
\end{tikzcd}$$

the canonical map

$$j^* g_* \to j^* g_* j'_* j'^* \simeq j^* j_* g'_* j'^* \to g'_* j'^*$$

is an equivalence.

(ii) $j_!$ satisfies base-change: For every cocartesian diagram as above, the canonical map

$$j_! g^* \to j_! g^* j'^* j'_! \simeq j_! g j^* g'^* j'_! \to g'^* j'_!$$

is an equivalence.

(iii) $j_!$ satisfies projection formula: For every $M \in D_{\square}(\mathbb{Z}[T], \mathbb{Z})$ and $N \in D_{\square}(\mathbb{Z}[T])$, the canonical map

$$j_!(j^*(M) \otimes N) \to j_!(j^*(M) \otimes j^* j_!(N)) \simeq j_! j^*(M \otimes j_!(N)) \to M \otimes j_!(N)$$

is an equivalence.

The idea of the proof is to take $\mathcal{C} \simeq D_{\square}(\mathbb{Z}[T], \mathbb{Z})$ and use that j^* is a Bousfield localization that we want to fit into the stable recollement from Lemma 18. To do so, we need an appropriate idempotent algebra-object, and this will turn out to be some kind of "circle at infinity" corresponding to $\mathbb{Z}((T^{-1}))$. We interpret this as "removing" the circle at infinity from the plane.

Proof. Note that $\mathbb{Z}((T^{-1}))$ sits in a sequence

$$\mathbb{Z}[[U]] \otimes \mathbb{Z}[T] \xrightarrow{(UT-1)} \mathbb{Z}[[U]] \otimes \mathbb{Z}[T] \longrightarrow \mathbb{Z}((T^{-1}))$$

from which we should be able to see that $\mathbb{Z}((T^{-1}))$ is idempotent[2]. Thus we go into Lemma 18 with $A = \mathbb{Z}((T^{-1}))$. The goal is to identify the right-hand side of the diagram. According to Lemma 19, it suffices to show that the j^*-local morphisms are exactly the morphisms f in $D_{\square}(\mathbb{Z}[T], \mathbb{Z})$ with fiber admitting an A-module structure.

Note that the j^*-local morphisms are generated by the unit of the adjunction (j^*, j_*) applied to the compact generators, i.e. the maps

$$\mathbb{Z}^I \otimes \mathbb{Z}[T] \to j_* j^*(\mathbb{Z}^I \otimes \mathbb{Z}[T]) \simeq \mathbb{Z}[T]^I.$$

But we have cocartesian squares like these:

$$\begin{tikzcd}
\mathbb{Z}^I \otimes \mathbb{Z}[T] \arrow[r] \arrow[d] & \mathbb{Z}[T]^I \arrow[r] \arrow[d] & F_1[1] \arrow[d, "\sim"] \\
\mathbb{Z}[[T^{-1}]]^I \otimes \mathbb{Z}[T] \arrow[r] \arrow[d] & \mathbb{Z}((T^{-1}))^I \arrow[r] \arrow[d] & F_2[1] \arrow[d] \\
(T^{-1}\mathbb{Z}[[T^{-1}]])^I \arrow[r, "\sim"'] & (\mathbb{Z}((T^{-1}))/\mathbb{Z}[T])^I \arrow[r] & 0
\end{tikzcd}$$

so we see that $F_1 \simeq F_2$ admits the structure of an A-module.

Conversely, it suffices to show that $j^*(M) = 0$ for any M with a A-module structure, for in that case $j^*(\mathrm{fib}(f)) = 0$ for any morphism f in $D_{\square}(A[T], A)$ with fiber admitting the structure of an A-module. But for such M,

$$j^*(i_*(M)) \simeq j^*(i_* i^* i_*(M)) = j^*(i_*(M) \otimes A) \simeq j^*(M) \otimes j^*(A),$$

so it suffices to show that $j^*(A) \simeq 0$. To that end, applying j^* to ... yields a sequence

$$\mathbb{Z}[[U,T]] \xrightarrow{(1-UT)} \mathbb{Z}[[U,T]] \longrightarrow j^*(A)$$

but the element $(UT-1)$ is a unit (with inverse $\sum (UT)^i$) so the map is an equivalence; whence $j^*(A) = 0$.

For (i), we may use Lemma 18 to compute

$$\begin{aligned}
j_* j^*(\underline{\mathrm{Hom}}(\mathbb{Z}^I \otimes \mathbb{Z}[T], M))(*) &\simeq \underline{\mathrm{Hom}}(F, \underline{\mathrm{Hom}}(\mathbb{Z}^I \otimes \mathbb{Z}[T], M))(*) \\
&\simeq \underline{\mathrm{Hom}}(F \otimes \mathbb{Z}^I \otimes \mathbb{Z}[T], M)(*) \\
&\simeq \underline{\mathrm{Hom}}(\mathbb{Z}^I \otimes \mathbb{Z}[T], \underline{\mathrm{Hom}}(F, M))(*) \\
&\simeq \underline{\mathrm{Hom}}(\mathbb{Z}^I \otimes \mathbb{Z}[T], j_* j^*(M))(*) \\
&\simeq j_* j^*(M)(S),
\end{aligned}$$

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[2] how exactly?

for every profinite set S (and I such that $(\mathbb{Z}[T],\mathbb{Z})_\square(S) \simeq \mathbb{Z}^I \otimes \mathbb{Z}[T]$). As j_* is fully faithful, we find

$$\begin{aligned}(\underline{\mathrm{Hom}}(\mathbb{Z}^I \otimes \mathbb{Z}[T], M) \otimes_{(\mathbb{Z}[T],\mathbb{Z})_\square} (\mathbb{Z}[T],\mathbb{Z}[T])_\square)(*) &\simeq j^*(\underline{\mathrm{Hom}}(\mathbb{Z}^I \otimes \mathbb{Z}[T], M))(*) \\ &\simeq j^*(M)(S) \\ &\simeq (M \otimes_{(\mathbb{Z}[T],\mathbb{Z})_\square} (\mathbb{Z}[T],\mathbb{Z}[T])_\square)(S).\end{aligned}$$

Hence by [Sch20, Proposition 13.9], j satisfies is steady.

(ii) follows from (i) and "formally passing to right adjoints":

$$\mathrm{Map}(j_!g^*(-),-) \simeq \mathrm{Map}(-, g_*j^*(-)) \simeq \mathrm{Map}(-, j'^*g'_*(-)) \simeq \mathrm{Map}(g'^*j'_!(-),-).$$

For (iii), note that the projection formula becomes an equivalence when applying j^* by the triangle-identities, so its cofiber C becomes an A-module (as $j^*(C) \simeq 0$). But $j_!j^*(M) \otimes A \simeq 0$ by the following sequence

$$j_!j^*(M) \otimes A \longrightarrow M \otimes A \xrightarrow{\sim} M \otimes A \otimes A$$

coming form the stable recollement and idempotency of A, so tensoring the projection formula with A, both terms become 0. Hence, $C \simeq C \otimes A \simeq 0$, so the projection formula holds. □

Remark 21. We are supposed to think about $(\mathbb{Z}[T],\mathbb{Z})$ as the plane with a "circle at infinity". The stable recollement is removing the "circle at infinity", corresponding to $\mathbb{Z}((T^{-1}))$, and we are left with the "plane", corresponding to $(\mathbb{Z}[T],\mathbb{Z}[T])$.

Next, lets treat the case of a general open immersion.

Proposition 22. If $j' : (B, A^+) \to (B, B^+)$ is an open immersion, then j^* admits a left adjoint $j_!$. Furthermore,

(i) j'^* satisfies base-change,
(ii) $j'_!$ satisfies base-change,
(iii) $j'_!$ satisfies projection formula.

Proof. First, we will prove (i). In view of **??** we may reduce to $A \simeq \mathbb{Z}$ (left-cancellative). In this case, B^+ is of the form $\mathbb{Z}[T_1, \dots, T_n]$, and there is a cocartesian diagram

$$\begin{tikzcd}(B, B^+) & (B^+, B^+) \arrow[l] \\ (B, \mathbb{Z}) \arrow[u] \arrow[ur, phantom, "\text{cocart}"] & (B^+, \mathbb{Z}). \arrow[l] \arrow[u]\end{tikzcd}$$

As steadiness is preserved under arbitrary base-change, it suffices to treat the case of

$$(\mathbb{Z}[T_1, \dots, T_n], \mathbb{Z}) \to (\mathbb{Z}[T_1, \dots, T_n], \mathbb{Z}[T_1, \dots, T_n]).$$

But this inductively reduces to $j : (\mathbb{Z}[T],\mathbb{Z}) \to (\mathbb{Z}[T],\mathbb{Z}[T])$, which is Theorem 20.

For the existence of $j'_!$, it suffices to show that j'^* preserves all limits. Since j' is an open immersion, B^+ is generated over A^+ by finitely many elements, $x_1, \dots, x_n$, and therefore sit in a cocartesian square like this (here g is the map $T_i \mapsto x_i$):

$$\begin{tikzcd}
(B,B^+) & (\mathbb{Z}[T_1,\dots,T_n],\mathbb{Z}[T_1,\dots,T_n]) \arrow[l, "g'"'] \\
(B,A^+) \arrow[u, "j'"] & (\mathbb{Z}[T_1,\dots,T_n],\mathbb{Z}) \arrow[l, "g"] \arrow[u, "j"']
\end{tikzcd}$$

Since all rings are finitely generated (over $\mathbb{Z}$), we may assume that g is surjective, and thus a localization. This is preserved by pushouts, so g' is a localization as well. But from steadiness, the canonical map $j^*g_* \to g'_*j'^*$ is an equivalence, and since g'_* is fully faithful, it follows that j'^* preserves limits. Explicitly, the left adjoint $j'_!$ is given by $g^*j_!g'_*$:

$$\mathrm{Map}(-, j'^*(-)) \simeq \mathrm{Map}(g'_*(-), g'_*j'^*(-)) \xleftarrow{\sim} \mathrm{Map}(g'_*(-), j^*g_*(-)) \simeq \mathrm{Map}(g^*j_!g'_*(-), -).$$

(ii) follows from (i) and formally passing to right adjoints.

For (iii), note that both sides in the projection formula commutes with colimits, so we may reduce to the case of compact projective generators; namely $M \simeq A^+[S] \otimes B$ and $N \simeq B^+[S] \otimes B$. Using (i), the cocartesian diagrams above, and induction, we may reduce to $B \simeq B^+ \simeq \mathbb{Z}[T]$, and $A^+ \simeq \mathbb{Z}$, which has been established.

$\square$

Now [HM24, Proposition 3.3.3] applies:

Theorem 23. The functor $D_{\square} : \mathrm{Huber}^{\mathrm{ft}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ promotes to a six-functor formalism

$$D_{\square} : \mathrm{Corr}(\mathrm{Huber}^{\mathrm{ft,op}}) \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}),$$

and it restricts to a six-functor formalism

$$D_{\square} : \mathrm{Corr}(\mathrm{AffSch}^{\mathrm{ft}}) \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}).$$

It still remains to globalize six-functor formalism, for which we need to promote $\mathrm{Huber}^{\mathrm{ft,op}}$ to a site. This is the content of the next section.

Globalization

In this section we will extend the six-functor formalism constructed in the previous section to a six-functor formalism on $\mathrm{Sh}(\mathrm{Huber}^{\mathrm{ft,op}})$ for an appropriate subcanonical site.

Proposition 24. The site from **??** on $\mathcal{C}^{\mathrm{op}} = \mathrm{Huber}^{\mathrm{ft,op}}$ with L be generated by

$$(\mathbb{Z}[T], \mathbb{Z}) \longrightarrow (\mathbb{Z}[T], \mathbb{Z}[T]) \tag{2}$$

and

$$(\mathbb{Z}[T], Z) \longrightarrow (\mathbb{Z}[T, T^{-1}], \mathbb{Z}), \tag{3}$$

is subcanonical, and functor $D_{\square} : (\mathrm{Huber}^{\mathrm{ft,op}})^{\mathrm{op}} \simeq \mathrm{Huber} \to \mathrm{Cat}_\infty$ is a sheaf of categories. Furthermore, the fully faithful embedding of rings into Huber pairs promotes to a map of sites when equipping the former with the Zariski topology.

Proof. For the first assertion, it suffices to verify that the two generating morphisms are steady localizations, but this follow easily from **??**. The second assertion requires (possibly) some non-trivial commutative algebra (in spirit with showing that the structure sheaf is a sheaf when doing affine schemes). This will be omitted. The third assertion is the content of Lemma **??**.

For the final assertion, note that the Zariski-topology is the topology from **??** with $\mathcal{C} = \mathrm{Ring} = \mathrm{AffSch}^{\mathrm{op}}$ and L generated by $\mathbb{Z}[T] \to \mathbb{Z}[T, T^{-1}]$, and the inclusion of rings into Huber pairs preserves covering sieves. ☐

Remark 25. We say that $D_\square$ satisfies *Zariski descent.*

Now [HM24, Theorem 3.4.11] applies, and we have finally constructed the solid six-functor formalism on schemes of finite type.

Theorem 26. The six-functor formalism from Theorem 23 promotes to a six-functor formalism

$$D_\square : \mathrm{Corr}(\mathrm{Sh}(\mathrm{AffAdicSpaces}^{\mathrm{ft}}, E) \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$$

where the class E of !-able morphisms contains maps that are separated and of finite type. Furthermore, there is a map of geometric setups $(\mathrm{Sch}^{\mathrm{ft}}, E') \to \mathrm{Sh}(\mathrm{AffAdicSpaces}^{\mathrm{ft}}, E)$ where $E' = \{\text{separated and of finite type}\}$.

Proof. In [HM24, Theorem 3.4.11], the class E is described. It certainly contains the maps that are separated and of finite type, as these properties globalize form the affine case. The map of geometric setups is just the inclusion. ☐

Suave and prim

In this section we will investigate which maps are suave for the six-functor formalism

$$D_\square : \mathrm{Corr}(\mathrm{AffSch}^{\mathrm{ft}}) \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}}).$$

We recall the following properties of suave maps:

Proposition 27. Let D be a six-functor formalism on a geometric setup $(\mathcal{C}, E)$ and let $f : Y \to X$ be a map in E. The map f is suave if and only if the canonical map

$$p^* f^!(1) \longrightarrow q^! f^*(1) \simeq q^!(1)$$

is an equivalence, where $p, q : Y \times_X Y \to Y$ are the projections. In fact, it suffices to show the equivalence upon applying $\mathrm{Map}(1, \Delta^!(-))$, where $\Delta^! : Y \to Y \otimes_X Y$ is the diagonal. If f is suave, then

(i) $f^!$ satisfies base-change in the sense that the canonical map $g'^* f^! \to f'^! g^*$ is an equivalence, and

(ii) the canonical map $f^!(1) \otimes f^*(-) \to f^!(-)$ is an equivalence. In particular, $f^!$ preserves all colimits.

Similarly, the map f is prim if and only if the canonical "norm" map

$$f_! q_* \Delta_!(1) \longrightarrow f_*(1)$$

is an equivalence, which may also be checked upon applying $\mathrm{Map}(1, -)$.

Proof. See [HM24, Section 4.5]. □

Proposition 28. If $f : Y \to X$ is a proper map of schemes of finite type, then f is $D_{\square}$-proper. In particular, there is a map $f_! \to f_*$ which is an equivalence, and $f_* f^!(-) \simeq \underline{\mathrm{Hom}}(f_*(1), -)$.

Proof. The first assertion follows from construction. For the other claims, see [HM24, Lemma 4.6.4(ii)]. Finally, we identify

$$f_* f^!(-) \simeq f_* \underline{\mathrm{Hom}}(1, f^!(-)) \simeq \underline{\mathrm{Hom}}(f_!(1), -) \simeq \underline{\mathrm{Hom}}(f_*(1), -).$$

□

Proposition 29. If $f : Y \to X$ is a map of schemes of finite type that is of finite Tor-dimension, then f is suave. In particular, if f is smooth of relative dimension d, then f is suave, and in that case, there is an identification $f^!(1) \simeq \Omega^d_{Y/X}[d]$.

Remark 30. Mann has informed us that the converse is actually true: If $f : Y \to X$ is suave, then it is of finite Tor-dimension.

Proof. We may assume that $X \simeq \operatorname{Spec} A$ and $Y \simeq \operatorname{Spec} B$ are affine. As f is of finite type, we can write $B \simeq A[t_1, \dots, t_n]/I$, and then factor $A \to B$ as $A \to A[t_1, \dots, t_n] \to B$. Thus we have reduced $f : \mathbb{A}^n_A \to A$, and f a closed immersion (into some affine space). For the former case, we can reduce to $f : \mathbb{A}^1_A \to A$ (by induction), and by base-change to $f : \mathbb{A}^1_{\mathbb{Z}} \to \mathbb{Z}$.

Let $A = \mathbb{Z}$ and $B = \mathbb{Z}[T]$, and consider the diagram below.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{cocart}"] & B \arrow[d, "p"] \arrow[ddr, bend left, "="] & \\
B \arrow[r, "q"'] \arrow[drr, bend right, "="'] & B \otimes_A B \arrow[dr, "\Delta"'] & \\
& & B
\end{tikzcd}$$

We will establish the equivalence in Proposition 27.

We claim that $f^!(1) \simeq 1[1]$. Indeed, using the factorization $\mathbb{Z} \to (\mathbb{Z}[T], \mathbb{Z}) \xrightarrow{j} (\mathbb{Z}[T], \mathbb{Z}[T])$ and Theorem 20 to see that $f_!(1) \simeq j_!(1) \simeq j_! j^*(1) \simeq 1 \otimes F \simeq F$, where F is the fiber of $\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))$, and then identify

$$f_* f^!(1) \simeq \underline{\mathrm{Hom}}(f_!(1), 1) \simeq \mathrm{Fib}(\underline{\mathrm{Hom}}(\mathbb{Z}((T^{-1})), \mathbb{Z}) \to \underline{\mathrm{Hom}}(\mathbb{Z}[T], \mathbb{Z})) \simeq \mathrm{Fib}(0 \to \mathbb{Z}) \simeq \mathbb{Z}[1] \simeq 1[1],$$

and hence by fully faithfulness, $f^!(1) \simeq f^*(1[1]) \simeq 1[1]$. Similarly, the map $q : \mathbb{Z}[T_1] \to \mathbb{Z}[T_1, T_2] \simeq (\mathbb{Z}[T_1])[T_2]$ satisfies $q^!(1) \simeq 1[1]$, so we can compute that $p^* f^!(1) \simeq p^*(1[1]) \simeq 1[1] \simeq q^!(1)$, yielding suaveness for the affine line $\mathbb{Z} \to \mathbb{Z}[T]$.

Next, assume that $f : A \to B$ is surjective (closed immersion). In this case, $f_* \simeq f_!$ and $f^!$ is given by $\underline{\mathrm{Hom}}(f_*(1), -)$. From [Lur17, Proposition 7.2.4.23], we see that f is of finite Tor-dimension if and only if B is perfect as an A-module. But being perfect is equivalent to being dualizable, which in turn is equivalent to $\underline{\mathrm{Hom}}(B, -)$ being linear. This shows that suaveness implies finite Tor-dimension due to Proposition 27.(ii). [3]

---

[3] I am not sure how to go about the other direction.

For the final assertion, assume that $f : A \to B$ is smooth of relative dimension d, or, equivalently that the diagonal $\Delta : B \otimes_A B \to B$ is a regular closed immersion of codimension d. Both f and Δ are of finite tor, and hence suave, so using the diagram above, we may formally identify

$$\begin{aligned} f^!(1) &= \Delta^! q^! f^!(1) = \Delta^!(q^!(1) \otimes q^* f^!(!)) \\ &= \Delta^!(p^* f^!(1) \otimes q^* f^!(1)) \\ &= \Delta^*(p^* f^!(1) \otimes q^* f^!(1)) \otimes \Delta^!(1) \\ &= (\Delta^* p^* f^!(1) \otimes \Delta^* q^* f^!(1)) \otimes \Delta^1(1) \\ &= f^!(1) \otimes f^!(1) \otimes \Delta^!(1). \end{aligned}$$

Using a the Koszul resolution, we observe that $f^!(1)$ is invertible (this holds more generally for f a local complete intersection), and whence by Lemma 31,

$$f^!(1) = \Delta^!(1)^\vee \simeq \underline{\mathrm{Hom}}(\Delta_*(1), 1)^\vee \simeq (\bigwedge^d (I/I^2)_B^\vee[-d])^\vee \simeq \Omega^d_{B/A}[d]$$

(where I is the kernel of Δ.) ☐

Lemma 31. If $g : \Delta : R \to S$ is a regular closed immersion of codimension d and with kernel I, then $\underline{\mathrm{Hom}}(g_*(R), S) \simeq \bigwedge_B^d (I/I^2)^\vee[-d]$.

Proof sketch. The Koszul complex resolution of $f_*(1)$ gives a resolution of $\underline{\mathrm{Hom}}(f_*(1), 1)$ where the cohomology vanish in all degrees but the top-degree (degree d) where it is $\bigwedge^d (I/I^2)^\vee$. Shifting down yields the claim. ☐

Remark 32. The computation of $f^!(1)$ above is apparently known as "Deligne's trick."

Discrete and pseudocoherent objects

In this section we will introduce the notions of "discrete" and "pseudocoherent" modules.

Definition 33. For a ring A, we define $D_\delta(A) \subset D_\square(A)$ to be the full subcategory on discrete objects.

Recall that an object $X \in \mathrm{Cond}(\mathcal{C})$ is called discrete if (and only if) for every profinite set $S = \lim_i S_i$, the canonical map $\mathrm{colim}_i X(S_i) \to X(S)$ is an equivalence. In the case of modules, this is equivalent to being a colimit of copies of A.

Remark 34. If $X \simeq \mathrm{Spec}(A)$, then $D_\delta(A)$ coincides with the derived ∞-category of the abelian category of A-modules.

Definition 35. For a ring A, we say that a module $M \in D_\square(A)$ is *pseudo-coherent* if it satisfies the following property: For every family $(N_i)_{i \in I}$ in $D_\square(A)$ such that $\pi_n(N_i) = 0$ for $n \gg 0$, for all $i \in I$ the canonical map

$$\bigoplus_{i \in I} \underline{\mathrm{Hom}}(M, N_i) \to \underline{\mathrm{Hom}}(M, \bigoplus_{i \in I} N_i)$$

is an equivalence. Denote by $D_{\mathrm{pc}}(A) \subset D_\square(A)$ to be the full subcategory on pseudocoherent modules.

So a pseudocoherent object is a corepresentable functor that preserves direct sums of $\leqslant n$-coconnective objects.

Remark 36. In [Sch25], Scholze defines an object $M \in D(A)$ to be pseudocoherent if for every n there is a perfect complex K_n and a map $K_n \to M$ whose cone sits in homological degress $\geqslant n$.

Remark 37. Lurie calls these modules "almost perfect" in [Lur17, Section 7.2.4], but this is apparently misleading since it has nothing to do with "almost mathematics."

The following result is the main claim of this section:

Proposition 38. The assignments $A \mapsto D_\delta(A)$ and $A \mapsto D_{\mathrm{pc}}(A)$ promote to subfunctors $D_\delta, D_{\mathrm{pc}} : \mathrm{AffSch}^{\mathrm{ft,op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$ of $D_\square : \mathrm{AffSch}^{\mathrm{ft,op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_{\mathrm{st}})$, and in fact subsheaves. Therefore, they globalize to functors on $\mathrm{Sch}^{\mathrm{ft,op}}$.

Furthermore, if $f : Y \to X$ is a map of schemes of finite type, then

(i) $f_*(D_\delta(Y)) \subset D_\delta(X)$,
(ii) $f_!(D_{\mathrm{pc}}(Y)) \subset D_{\mathrm{pc}}(X)$, provided that f is !-able.

We postpone the proof to the end of this section. First, we will get to know the discrete objects:

Lemma 39. If $Y \in D_\square(A)$ is discrete and $S = \lim_i S_i$ is a profinite set, then the canonical map

$$\operatorname*{colim}_i \underline{\mathrm{Hom}}(A_\square[S_i], Y) \to \underline{\mathrm{Hom}}(A_\square[S], Y)$$

is an equivalence.

Proof. See [Asg24, Theorem B]. □

Proposition 40. If A is a ring, then $P \in D_\square(A)$ is discrete if and only if the canonical map

$$(\underline{\mathrm{Hom}}(A_\square[S], M) \otimes P)(*) \longrightarrow \underline{\mathrm{Hom}}(A_\square[S], M \otimes P)(*) \simeq (M \otimes P)(S) \tag{4}$$

is an equivalence for all profinite sets S and modules $M \in D_\square(A)$.

Proof. Let $S = \lim_i S_i$ be a profinite set, and assume $P \in D_\square(A)$ satisfies Eq. (4). Use Lemma 39 to compute

$$\begin{aligned} P(S) \simeq (\underline{\mathrm{Hom}}(A_\square[S], A) \otimes P)(*) &\simeq (\underline{\mathrm{Hom}}(\lim_i A[S_i], A) \otimes P)(*) \\ &\simeq (\operatorname*{colim}_i \underline{\mathrm{Hom}}(A[S_i], A) \otimes P)(*) \\ &\simeq \operatorname*{colim}_i (\underline{\mathrm{Hom}}(A[S_i], A) \otimes P)(*) \\ &\simeq \operatorname*{colim}_i (\underline{\mathrm{Hom}}(A_\square[S_i], A) \otimes P)(*) \\ &\simeq \operatorname*{colim}_i P(S_i). \end{aligned}$$

Conversely, any discrete object is a colimit of copies of A, but A evidently satisfies Eq. (4), which conveniently commutes with colimits. □

Remark 41. The statement in Proposition 40 holds more generally for every Huber pair (A, A^+) by a similar argument.

Remark 42. Modules satisfying Eq. (4) are called *nuclear*. In fact, for such modules, the equivalence can be strengthened to not just hold on global sections, but as sheaves on $\mathrm{ProfSet}^{\mathrm{light}}$: If $R = (R^{\triangleright}, D(R))$ is an analytic ring, then we denote by $D(R)^{\mathrm{nuc}}$ the full subcategory of $D(R)$ spanned on those objects $P \in D(R)$ for which the canonical map

$$(\underline{\mathrm{Hom}}(R[S], M) \otimes P)(*) \longrightarrow \underline{\mathrm{Hom}}(R[S], M \otimes P)(*),$$

is an equivalence for all profinite sets S and modules M. A tensor-Hom identification shows that $f : R \to R'$ is a map such that R' is nuclear as an R-module, then the canonical map

$$\underline{\mathrm{Hom}}(R[S], M) \otimes R' \longrightarrow \underline{\mathrm{Hom}}(R'[S], M \otimes R')$$

is an equivalence. That is, $f^*\underline{\mathrm{Hom}}(R[S], M) \simeq \underline{\mathrm{Hom}}(R'[S], f^*(M))$.

We can use this remark to get a criterion for checking when f_* preserves nuclear objects.

Lemma 43. If $f : R \to R'$ is a map of analytic rings, then $f_* : D(R')^{\mathrm{nuc}} \to D(R)$ factors through $D(R)^{\mathrm{nuc}}$.

Proof. By projection formula, we have

$$\begin{aligned}
\underline{\mathrm{Hom}}(R[S], M) \otimes f_*(P) &\simeq f_*(f^*\underline{\mathrm{Hom}}(R[S], M) \otimes P) \\
&\simeq f_*(\underline{\mathrm{Hom}}(R'[S], f^*(M)) \otimes P) && \text{by nuclearity of } R' \\
&\simeq f_*(\underline{\mathrm{Hom}}(f^*R[S], f^*(M) \otimes P)) && \text{by nuclearity of } P \\
&\simeq \underline{\mathrm{Hom}}(R[S], f_*(f^*(M) \otimes P)) \\
&\simeq \underline{\mathrm{Hom}}(R[S], M \otimes f_*(P)),
\end{aligned}$$

whence the claim. □

Next, we will try to understand the pseudocoherent objects. We will first use the "internal adjunction" formulas,

$$f_*\underline{\mathrm{Hom}}(f^*(-), -) \simeq \underline{\mathrm{Hom}}(-, f_*(-)) \quad \text{and} \quad \underline{\mathrm{Hom}}(f^*(-), -) \simeq \underline{\mathrm{Hom}}(-, f_*(-)),$$

to cook up two criteria for checking whether f^* (resp. $f^!$) takes pseudocoherent objects to pseudocoherent objects. This is the content of the next two lemmas:

Lemma 44. If $f : A \to B$ be a map such that

1. f_* is conservative,
2. f_* preserves filtered colimits,
3. for every n, there is an m such that $f_* : D_{\leqslant n,\square}(B) \to D_{\square}(A)$ factors through $D_{\leqslant m,\square}(A)$, and

then $f^* : D_{\mathrm{pc}}(A) \to D_{\square}(B)$ factors through $D_{\mathrm{pc}}(B)$.

Note that the condition of commuting with filtered colimits implies that f_* commutes with direct sums, as a direct sum is a filtered colimit of finite direct sums, and finite colimits can be expressed as (finite) limits in the stable setting.

Proof of Lemma 44. Let M be in $D_{\mathrm{pc}}(A)$ and let $(N_i)_I$ be a family of objects in $D_{\leqslant,n\square}(B)$. To check if the canonical map

$$\bigoplus_I \underline{\mathrm{Hom}}(f^*(M), N_i) \longrightarrow \underline{\mathrm{Hom}}(f^*(M), \bigoplus_I N_i)$$

is an equivalence after applying f_*. But using that f_* preserves direct sums and the internal adjunction formula, this reduces to showing that

$$\bigoplus_I \underline{\mathrm{Hom}}(M, f_*(N_i)) \longrightarrow \underline{\mathrm{Hom}}(M, \bigoplus_I f_*(N_i))$$

is an equivalence. But by assumption, $(f^*(N_i))_I$ is a family of objects in $D_{\leqslant m,\square}(B)$ for some m, and thus the above map is an equivalence by pseudocoherence of M. ☐

Lemma 45. If $f : A \to B$ be a map such that

1. $f^!$ preserves filtered colimits,
2. f_* preserves filtered colimits, and
3. for every n, there is an m such that $f^! : D_{\leqslant n,\square}(A) \to D_\square(B)$ factors through $D_{\leqslant m,\square}(B)$,

then $f_! : D_{\mathrm{pc}}(B) \to D_\square(A)$ factors through $D_{\mathrm{pc}}(A)$.

Proof. If M be in $D_{\mathrm{pc}}(B)$ Let $(N_i)_I$ be a family of objects in $D_{\leqslant,n\square}(A)$, then the bottom arrow in below diagram is an equivalence, and therefore $f_!(M)$ is pseudocoherent:

$$\begin{tikzcd}
\bigoplus_I \underline{\mathrm{Hom}}(f_!(M), N_i) \arrow[r] \arrow[d, "\sim"'] & \underline{\mathrm{Hom}}(f_!(M), \bigoplus_I N_i) \arrow[d, "\sim"] \\
\bigoplus_I f_*\underline{\mathrm{Hom}}(M, f^!(N_i)) \arrow[r] \arrow[d, "\sim"'] & f_*\underline{\mathrm{Hom}}(M, f^!(\bigoplus_I N_i)) \arrow[d, "\sim"] \\
f_* \bigoplus_I \underline{\mathrm{Hom}}(M, f^!(N_i)) \arrow[r] & f_*\underline{\mathrm{Hom}}(M, \bigoplus_I f^!(N_i))
\end{tikzcd}$$

☐

As in the previous section, it will reduce to the case of the affine line $\mathbb{Z} \to \mathbb{Z}[T]$ and a closed $A \to B$. To that end, we have

Lemma 46. If f denotes the map $\mathbb{Z} \to \mathbb{Z}[T]$, then f_* preserves filtered colimits and f^* is t-exact; in particular, $f^! : D_{\leqslant n,\square}(A) \to D_\square(B)$ factors through $D_{\leqslant n+1,\square}(B)$.

Proof. For the first assertion, use the factorization $\mathbb{Z} \xrightarrow{p} (\mathbb{Z}[T], \mathbb{Z}) \xrightarrow{j} \mathbb{Z}[T]$ of f and Theorem 20 to compute

$$f_*(-) \simeq p_* j_* j^* j_!(-) \simeq p_* \underline{\mathrm{Hom}}(F, j_!(-)).$$

As $F \simeq T^{-1}\mathbb{Z}[[T^{-1}]] \simeq \prod_{\mathbb{N}} \mathbb{Z}$ is a compact (projective) and $p_* \simeq p_!$ and $j_!$ are both left adjoints, the functor $f_*(-)$. preserves filtered colimits.

For the second assertion, we have the formula $f^*(-) \simeq \underline{\mathrm{Hom}}(F, -)$ where $F = T^{-1}\mathbb{Z}[[T^{-1}]]$. As F is (compact) projective, $f^*(-)$ is t-exact. In particular, $f^!(-) \simeq f^*(-) \otimes f^!(1) \simeq f^*(-) \otimes 1[1] \simeq f^*(-)[1]$ shifts the connective part by 1. ☐

Lemma 47. If $f : A \to B$ be a surjection (a closed immersion), then $f^!$ commutes with filtered colimits, f_* is conservative and preserves all colimits, and $f^! : D_{\leqslant n,\square}(A) \to D_\square(B)$ factors through $D_{\leqslant n,\square}(B)$.

Proof. Conservativity for f_* actually holds for affine morphisms and can be checked on stalks. As f is closed, it is proper, si we have $f_* f^!(-) \simeq \underline{\mathrm{Hom}}(f_*(1), -)$ and $f_* \simeq f_!$ shows that $f^!(-)$ commutes with filtered colimit, and that f_* which preserves all colimits. For the final assertion, note that $f_* f^!(-) \simeq \underline{\mathrm{Hom}}(f_*(1), -)$, which restricts down to coconnective parts. $\square$

Finally, we will provide justify Proposition 38.

Proof of Proposition 38. To show that the assignments $A \mapsto D_\delta(A)$ promotes to a functor, we must show that $f^* : D_\delta(A) \to D_\square(B)$ factors through $D_\delta(B)$ for every map $f : A \to B$. This is indeed the case; using the projection formula, we compute

$$\begin{aligned} f_*(\underline{\mathrm{Hom}}(B[S], M) \otimes f^*(P))(*) &\simeq (f_* \underline{\mathrm{Hom}}(B[S], M) \otimes P)(*) \\ &\simeq (\underline{\mathrm{Hom}}(A[S], f_*(M)) \otimes P)(*) \\ &\simeq (f_*(M) \otimes P)(S) && \text{by nuclearity} \\ &\simeq f_*(M \otimes f^*(P))(S), \end{aligned}$$

so $f^*(P)$ is nuclear (since the forgetful functor f_* is conservative). Similarly, we must show that f^* preserves pseudocoherent objects. we reduce to the claim for $f : \mathbb{Z} \to \mathbb{Z}[T]$ and $A \to B \simeq A/I$ a closed immersion. For $f : \mathbb{Z} \to \mathbb{Z}[T]$, it suffices to show that $f_* : D_{\leqslant n}(\mathbb{Z}[T]) \to D_\square(\mathbb{Z})$ factors through $D_{\leqslant m\square}(\mathbb{Z})$ for some m. For a closed immersion, this is just Lemma 44 and 47.

To show they are subsheaves, it suffices to show they are sheaves themselves. This follows by a similar argument as in **??**, noting that conservativity is a right-cancelable property.

For (i), we reduce to the claim for $f : \mathbb{Z} \to \mathbb{Z}[T]$ and $A \to B \simeq A/I$ a closed immersion. For $f : \mathbb{Z} \to \mathbb{Z}[T]$, note that $f_*(\mathbb{Z}[T]) \simeq \bigoplus_{\mathbb{N}} \mathbb{Z}$ (i.e. $\mathbb{Z}[T]$ as a $\mathbb{Z}$-module) which is discrete, or equivalently, nuclear by Proposition 40. so the claim follows from Lemma 43. For the closed immersion, note that $f_*(B)$ is discrete since it admits a free resolution (the Koszul complex), so it is also can be expressed as a colimit of copies of A. Again, the claim follows from Lemma 43.

For (ii), we reduce to the claim for $f : \mathbb{Z} \to \mathbb{Z}[T]$ and $A \to B$ a closed immersion. The case of the affine line follows from Lemma 45, 46, and 47, while the case of the closed immersion follows from Lemma 47 and 47. $\square$

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