Problem 1.

1. the map $\varphi : F \to F' = i(F')$ is a sheafification iff the induced maps on stalks $\varphi_x : F_x \to F'_x$ are bijections for all $x \in X$. We proceed with characterizing these stalks[1]: Consider the map $f : F_x \to A$ given by $f(s_x) = s(x)$ for $s_x \in F_x$. This is well-defined; given $(U,s) \sim (V,t)$ representing s_x, we find $x \in W \subset U \cap V$ s.t. $s_{|W} = t_{|W}$, so $s(x) = t(x)$. It is injective; given $s_x = [U,s], t_x = [V,t] \in F_x$ s.t. $s(x) = f(s_x) = f(t_x) = t(x)$, we get $s = t$ as this is the presheaf of constant functions, so in particular $(U,s) \sim (V,t)$, and thus $s_x = t_x$. It is clearly also surjective, as for given $a \in A$, we just take $s : U \to A$, $u \mapsto a$ and then $s_x = (U,a)$ maps to $s(x) = a$. Thus $F_x \cong A$.

   Similarly, we show $F_x \cong A$: Construct the same map $f : F'_x \to A$ given by $f(s_x) = s(x)$. Well-definedness and surjectiveness follow from exactly the same arguments used above. For injection: given $s_x = [U,s], t_x = [V,t] \in F_x$ s.t. $s(x) = f(s_x) = f(t_x) = t(x)$, we find open subsets $x \in U_x \subset U$, $x \in V_x \subset V$ so $s_{|U_x}$ and $t_{|V_x}$ are constant. In particular, $s_{|U_x} = s(x) = t(x) = t_{|V_x}$, so there is an x-ngbh $W = U_x \cap V_x \subset U \cap V$ s.t. $s_{|W} = t_{|W}$ - i.e. $s_x = t_x$.

   Thus $F'_x \cong A$, so the induced map on stalks $\varphi_x : F_x \to F'_x$ is a bijection

2. We claim that X being irreducible implies any locally constant function $s : U \to A$ is a constant function, which would make $\varphi_U : F(U) \to F'(U)$ a bijection of sets, and thus F an isomorphism. Recall from Problem 5 on the previous week's exercise sheet that X being irreducible implies any open $U \subset X$ is connected. Assume that $s : U \to A$ takes more than one value in A. Let $a \in A$ be a value of s. For any $x \in U$ we find a ngbh $U_i \subset U$ containing x such that $s_{|U_x}$ is constant. Let

$$V_1 = \bigcup_{s_{|U_x} = a \in A} U_x \text{ and } V_2 = \bigcup_{s_{|U_x} \neq a \in A} U_x.$$

   We claim that V_1 and V_2 constitute an open partition of U. We clearly have $U = \bigcup_{x \in U} U_x = V_1 \cup V_2$. Both V_1 and V_2 are open as they are a union of opens. Furthermore, $V_1 \cap V_2 = \varnothing$ by construction, and by assumption, they are both non-empty. But this contradicts the fact that U is connected, so all sections $s \in F(U)$ must be constant.

Problem 2. We claim that if y specializes to x then every open x-ngbh contains y. Indeed, let U be any such ngbh, and assume that $y \notin U$. Then U^c contains y and is closed. But as $x \in \overline{\{y\}} = \bigcap_{V \subset X \text{ closed}, x \in V} V$ we see that x lies on every closed subset containing y. In particular, it lies in U^c which is a contradiction. Now, by the universal property of colimits, we find a "co-specialization map" $\varphi^F : \mathrm{colim}_{x \in U \subset X} F(U) = F_x \to \mathrm{colim}_{y \in U \subset X} F(U) = F_y$. Explicitly, for $s_x \in F_x$, find a representative (U,s) and then map it to $s_y = [U,s]$ in F_y. This is well-defined as every open x-ngbh is also an open y-ngbh, and if (U,s) and (V,s) are two representatives for s_x then we certainly also have $(U,s) \sim (V,t)$ regarded as y-ngbhs. To prove naturality, we need the induced diagram to the left below in Set to commute for $\varphi : F \to G$ in $\mathrm{Sh}(X)$:

$$\begin{tikzcd} F_x \arrow[r, "\varphi^F"] \arrow[d, "\varphi_x"] & F_y \arrow[d, "\varphi_y"] \\ G_x \arrow[r, "\varphi^G"] & G_y. \end{tikzcd} \quad , \quad \begin{tikzcd} F(U) \arrow[r, "\varphi_U"] \arrow[d] & G(U) \arrow[d] \\ F_x \arrow[r, "\varphi_x"] & G_x. \end{tikzcd}$$

Indeed, for $s_x \in F_x$, take representative (U,s). Going top and down takes s_x to s_y represented by (U,s) which goes to $\varphi_U(s)_y$ represented by $(U, \varphi_U(s))$ (here we used something like the right diagram above). On the other hand, going down and bottom takes s_x to $\varphi_U(s)_x$ represented $(U, \varphi_U(s))$ (above right diagram) and then to $\varphi_U(s)_y$ represented by $(U, \varphi_U(s))$. Thus the desirede left diagram above commutes.

[1] Intuitively with the slogan "eventually equal implies equal on stalks", it is evident that both F_x and F'_x should be isomorphic to A, but this is an assignment counting towards a grade, so we should probably check it formally.