Problem 1. For every open $x \in U \subset |X|$ we have a map $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ and thus a map $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_X(U))$. In particular, for affine open $x \in U \subset |X|$, a map of schemes $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_X(U)) \cong U \to X$ where the last map is an open immersion. We show this map is independent of choice of affine open $U \ni x$, thus supplying a map $j_{X,x} : \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$: Let $x \in U, V \subset |X|$ be affine opens, and find an affine open $x \in W \subset U \cap V$. Then we get commutative diagrams

$$\begin{array}{ccccc} \mathcal{O}_X(U) & \xrightarrow{\mathrm{res}^U_W} & \mathcal{O}_X(W) & \xleftarrow{\mathrm{res}^V_W} & \mathcal{O}_X(V) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{O}_{X,x} & & \end{array} \quad \xrightarrow{\mathrm{Spec}(-)} \quad \begin{array}{ccccc} & & \mathrm{Spec}(\mathcal{O}_{X,x}) & & \\ & \swarrow & \downarrow & \searrow & \\ U & \longleftarrow & W & \longrightarrow & V. \end{array}$$

But from the following commutative diagram of open immersions

$$\begin{array}{ccccc} U & \longleftarrow & W & \longrightarrow & V \\ & \searrow & \downarrow & \swarrow & \\ & & X. & & \end{array}$$

The map $W \to U \to X$ are open immersions, so their composition is the open immersion $W \to X$. Similarly for the maps $W \to V \to X$. As open immersions are unique, we find the maps $\mathrm{Spec}(\mathcal{O}_{X,x} \to U \to X$ and $\mathrm{Spec}(\mathcal{O}_{X,x} \to V \to X$ are equal, and hence $\mathrm{Spec}(O_{X,x}) \to U \to X$ is independent of choice of U.

We show this map is natural: Let $f = (p, \phi) : X \to Y$ be a map of schemes. We need to show that there is a map making the below diagram to the left commute:

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{X,x}) & \xrightarrow{j_{X,x}} & X \\ {\scriptstyle \exists!}\downarrow & & \downarrow{\scriptstyle f=(p,\phi)} \\ \mathrm{Spec}(\mathcal{O}_{Y,p(x)}) & \xrightarrow[j_{Y,p(x)}]{} & Y, \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{j_U} & X \\ {\scriptstyle \exists!}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j_V} & Y. \end{array}$$

To this extend, use Lemma 5.13 to find an affine open $U \xrightarrow{j_U} X$ and $V \xrightarrow{j_V} Y$ such that $x \in U$, $p(x) \in V$ and a commutative diagram like the one above to the right. Using that U is contained in $p^{-1}(V)$ (by construction of U, see the proof of lemma 5.13) we get the following commutative diagram on stalks:

$$\begin{array}{ccccc} \mathcal{O}_Y(V) & \xrightarrow{\phi_U} & p_*\mathcal{O}_X(V) = \mathcal{O}_X(p^{-1}(V)) & \xrightarrow{\mathrm{res}^{p^{-1}(U)}_V} & \mathcal{O}_X(U) \\ \downarrow & & & \searrow & \downarrow \\ \mathcal{O}_{Y,p(x)} & & \xrightarrow{\phi_x} & & \mathcal{O}_{X,x}. \end{array}$$

So, by taking $\mathrm{Spec}(-)$ and using that U and V are affine opens, we find the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & U & \longrightarrow & X \\ {\scriptstyle \mathrm{Spec}(\phi_x)}\downarrow & & \downarrow & & \downarrow{\scriptstyle f=(p,\phi)} \\ \mathrm{Spec}(\mathcal{O}_{Y,p(x)}) & \longrightarrow & V & \longrightarrow & Y, \end{array}$$

which is what we wanted.

Finally, we show that the underlying topological image of $j_{X,x} : \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ is the points $\eta \in |X|$ s.t. $x \in \overline{\{\eta\}}$. First, observe that $\{\eta \in |X| | x \in \overline{\{\eta\}}\} = \bigcap_{x \in U \subset |X| \text{ open}} U$; indeed, assume $x \in \overline{\{\eta\}}$ and assume $U \subset |X|$ is an open x-ngbh not containing η. Then U^c is closed, contains η, and doesn't contain x. But as $x \in \overline{\{\eta\}} = \bigcap_{\eta \in V \subset |X| \text{closed}} V \subset U^c$ we get a contradiction. Now, assume η lies in every open x-ngbh. Assume $x \notin \overline{\{\eta\}}$. Then there is $\eta \in V \subset |X|$ closed such that $x \notin V$. But then V^c is open, contains x, and doesn't contain η - contradiction.

Let U be an affine open of $x \in |X|$. Then $j_{X,x}$ is given by the composition $\operatorname{Spec}(\mathcal{O}_{X,x}) \to U \to X$. Using the isomorphism of stalks $\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x}$ we see that the image of $j_{X,x}$ is the image of the induced map $A \to A_{\mathfrak{p}}$ where $\operatorname{Spec} A \equiv U$ and $\mathfrak{p} \lhd A$ correspond to $x \in U$. But it is known (problem set 1) that the image of this map is $\{\mathfrak{q} \in |\operatorname{Spec} A| | \mathfrak{q} \subset \mathfrak{p}\} = \{\eta \in U | x \in \overline{\{\eta\}}\} = \bigcap_{x \in V \subset U \text{open}} V = \bigcap_{x \in W \subset |X| \text{open}} U \cap W = \bigcap_{x \in W \subset |X| \text{open}} W$, where the last equality follows from U begin an open ngbh of x in $|X|$, as desired.

Problem 2.

1. Write $\mathcal{M}$ and $\mathcal{N}$ as $\tilde{M}$ and $\tilde{N}$ which is possible as X is affine. We want $\tilde{M} \xrightarrow{f} \tilde{N} \to 0$ to be exact as quasi-coherent $\mathcal{O}_X$-modules. As exactness can be checked on stalks, we want $\tilde{M}_x \to \tilde{N}_x \to 0$ to be exact as $\mathcal{O}_{X,x}$-modules for all $x \in X$. But this follows from (a version of) Nakayama's lemma[1]: As M is finitely generated R-module, $\bigoplus_1^n R \to M \to 0$ is exact. Taking $\tilde{(-)}$ we find $\bigoplus_1^n \mathcal{O}_X \to \tilde{M} \to 0$ is exact of quasi-coherent O_X-modules. Taking stalks thus amounts to $\bigoplus_1^n \mathcal{O}_{X,x} \to \tilde{M}_x \to 0$ being exact of $\mathcal{O}_{X,x}$-modules, and hence $\tilde{M}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module. Similarly, $\tilde{N}_x$ is finitely generated $O_{X,x}$-module. As $\mathcal{O}_{X,x}$ is a local ring, $\mathfrak{m}_x$ is its Jacobsen radical, and thus, as $f(x) : \tilde{M}_x/\mathfrak{m}_x\tilde{M}_x \to \tilde{N}_x/\mathfrak{m}_x\tilde{N}_x$ surjective, Nakayama implies that $f_x : \tilde{M}_x \to \tilde{N}_x$ is surjective for all $x \in |X|$ as desired.

2. We take $M = N = R = \mathbb{Z}/4\mathbb{Z}$. First, note that the only prime ideal of R is $(2) = \{0, 2\}$. So the open subsets of $|X|$ are $\varnothing$, $\{(2)\} = |X|$.

   Take the map $f : \mathcal{O}_X = \tilde{M} \to \mathcal{O}_X = \tilde{N}$ which on open $U \subset |X|$ (i.e. $U = X$) is multiplication by 2; $\mathbb{Z}/4\mathbb{Z} = \mathcal{O}_X(X) \to \mathbb{Z}/4\mathbb{Z} = \mathcal{O}_X(X)$ by multiplication by 2. There is only one point in $x \in |X|$ - the one corresponding to (2). So we only need to carry out calculations for this point. We have $\mathcal{O}_{X,x} = \operatorname{colim}_{x \in U \subset X} \mathcal{O}_X(U) = \mathcal{O}_X(X) = \mathbb{Z}/4\mathbb{Z}$ and thus $\mathfrak{m}_x = (2)$. It follows that the map

   $$(\mathbb{Z}/4\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z})/(2)(\mathbb{Z}/4\mathbb{Z}) = \mathcal{O}_X/\mathfrak{m}_x\mathcal{O}_X$$
   $$= \tilde{M}_x/\mathfrak{m}_x\tilde{M}_x \xrightarrow{\cdot 2} \tilde{N}_x/\mathfrak{m}_x\tilde{N}_x = (\mathbb{Z}/4\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})$$

   is the zero-map. This supplies a counter example, so the assertion is false.

[1] See `https://stacks.math.columbia.edu/tag/07RC` (6)