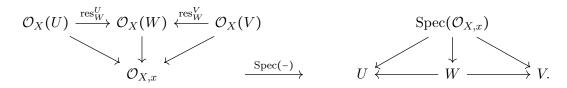
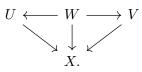
Problem 1. For every open $x \in U \subset |X|$ we have a map $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ and thus a map $\operatorname{Spec}(\mathcal{O}_{X,x}) \to \operatorname{Spec}(\mathcal{O}_X(U))$. In particular, for affine open $x \in U \subset |X|$, a map of schemes $\operatorname{Spec}(\mathcal{O}_{X,x}) \to \operatorname{Spec}(\mathcal{O}_X(U)) \cong U \to X$ where the last map is an open immersion. We show this map is independent of choice of affine open $U \ni x$, thus supplying a map $j_{X,x} : \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$: Let $x \in U, V \subset |X|$ be affine opens, and find an affine open $x \in W \subset U \cap V$. Then we get commutative diagrams



But from the following commutative diagram of open immersions

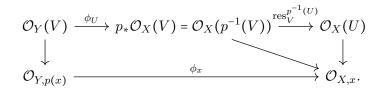


The map $W \to U \to X$ are open immersions, so their composition is the open immersion $W \to X$. Similarly for the maps $W \to V \to X$. As open immersions are unique, we find the maps $\operatorname{Spec}(\mathcal{O}_{X,x} \to U \to X \text{ and } \operatorname{Spec}(\mathcal{O}_{X,x} \to V \to X \text{ are equal, and hence } \operatorname{Spec}(\mathcal{O}_{X,x}) \to U \to X \text{ is independent of choice of } U.$

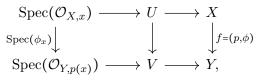
We show this map is natural: Let $f = (p, \phi) : X \to Y$ be a map of schemes. We need to show that there is a map making the below diagram to the left commute:

$\operatorname{Spec}(\mathcal{O}_{X,x}) \xrightarrow{j_{X,x}} \mathcal{I}$	X	$U \stackrel{j_U}{$	$\rightarrow X$
3!	$f=(p,\phi)$	Э!	$\int f$
$\operatorname{Spec}(\mathcal{O}_{Y,p(x)}) \xrightarrow{j_{Y,p(x)}} \mathcal{I}$	\downarrow Y,	$V \xrightarrow{j_V} -j_V$	\downarrow $\rightarrow Y.$

To this extend, use Lemma 5.13 to find an affine open $U \xrightarrow{j_U} X$ and $V \xrightarrow{j_V} Y$ such that $x \in U$, $p(x) \in V$ and a commutative diagram like the one above to the right. Using that U is contained in $p^{-1}(V)$ (by construction of U, see the proof of lemma 5.13) we get the following commutative diagram on stalks:



So, by taking Spec(-) and using that U and V are affine opens, we find the following commutative diagram:



which is what we wanted.

Finally, we show that the underlying topological image of $j_{X,x} : \operatorname{Spec}(\mathcal{O}_{X,x}) \to X$ is the points $\eta \in |X|$ s.t. $x \in \overline{\{\eta\}}$. First, observe that $\{\eta \in |X| | x \in \overline{\{\eta\}}\} = \bigcap_{x \in U \subset |X| \text{ open }} U$; indeed, assume $x \in \overline{\{\eta\}}$ and assume $U \subset |X|$ is an open x-ngbh not containing η . Then U^{c} is closed, contains η , and doesn't contain x. But as $x \in \overline{\{\eta\}} = \bigcap_{\eta \in V \subset |X| \text{ closed }} V \subset U^{\mathsf{c}}$ we get a contradiction. Now, assume η lies in every open x-ngbh. Assume $x \notin \overline{\{\eta\}}$. Then there is $\eta \in V \subset |X|$ closed such that $x \notin V$. But then V^{c} is open, contains x, and doesn't contain η - contradiction.

Let U be an affine open of $x \in |X|$. Then $j_{X,x}$ is given by the composition $\operatorname{Spec}(\mathcal{O}_{X,x}) \to U \to X$. Using the isomorphism of stalks $\mathcal{O}_{X,x} \cong \mathcal{O}_{U,x}$ we see that the image of $j_{X,x}$ is the image of the induced map $A \to A_{\mathfrak{p}}$ where $\operatorname{Spec} A \equiv U$ and $\mathfrak{p} \triangleleft A$ correspond to $x \in U$. But it is known (problem set 1) that the image of this map is $\{\mathfrak{q} \in |\operatorname{Spec} A || \mathfrak{q} \subset \mathfrak{p}\} = \{\eta \in U | x \in \overline{\{\eta\}}\} = \bigcap_{x \in V \subset U \text{open}} V = \bigcap_{x \in W \subset |X| \text{open}} U \cap W = \bigcap_{x \in W \subset |X| \text{open}} W$, where the last equality follows from U begin an open ngbh of x in |X|, as desired.

Problem 2.

- 1. Write \mathcal{M} and \mathcal{N} as \tilde{M} and \tilde{N} which is possible as X is affine. We want $\tilde{M} \xrightarrow{f} \tilde{N} \to 0$ to be exact as quasi-coherent \mathcal{O}_X -modules. As exactness can be checked on stalks, we want $\tilde{M}_x \to \tilde{N}_x \to 0$ to be exact as $\mathcal{O}_{X,x}$ -modules for all $x \in X$. But this follows from (a version of) Nakayama's lemma¹: As M is finitely generated R-module, $\bigoplus_1^n R \to M \to 0$ is exact. Taking $(\tilde{-})$ we find $\bigoplus_1^n \mathcal{O}_X \to \tilde{M} \to 0$ is exact of quasi-coherent \mathcal{O}_X -modules. Taking stalks thus amounts to $\bigoplus_1^n \mathcal{O}_{X,x} \to \tilde{M}_x \to 0$ being exact of $\mathcal{O}_{X,x}$ -modules, and hence \tilde{M}_x is a finitely generated $\mathcal{O}_{X,x}$ -module. Similarly, \tilde{N}_x is finitely generated $\mathcal{O}_{X,x}$ -module. As $\mathcal{O}_{X,x}$ is a local ring, \mathfrak{m}_x is its Jacobsen radical, and thus, as $f(x) : \tilde{M}_x/\mathfrak{m}_x \tilde{M}_x \to \tilde{N}_x/\mathfrak{m}_x \tilde{N}_x$ surjective, Nakayama implies that $f_x : \tilde{M}_x \to \tilde{N}_x$ is surjective for all $x \in |X|$ as desired.
- 2. We take $M = N = R = \mathbb{Z}/4\mathbb{Z}$. First, note that the only prime ideal of R is $(2) = \{0, 2\}$. So the open subsets of |X| are \emptyset , $\{(2)\} = |X|$.

Take the map $f : \mathcal{O}_X = \tilde{M} \to \mathcal{O}_X = \tilde{N}$ which on open $U \subset |X|$ (i.e. U = X) is multiplication by 2; $\mathbb{Z}/4\mathbb{Z} = \mathcal{O}_X(X) \to \mathbb{Z}/4\mathbb{Z} = \mathcal{O}_X(X)$ by multiplication by 2. There is only one point in $x \in |X|$ - the one corresponding to (2). So we only need to carry out calculations for this point. We have $\mathcal{O}_{X,x} = \operatorname{colim}_{x \in U \subset X} \mathcal{O}_X(U) = \mathcal{O}_X(X) = \mathbb{Z}/4\mathbb{Z}$ and thus $\mathfrak{m}_x = (2)$. It follows that the map

$$(\mathbb{Z}/4\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z})/(2)(\mathbb{Z}/4\mathbb{Z}) = \mathcal{O}_X/\mathfrak{m}_x\mathcal{O}_X$$
$$= \tilde{M}_x/\mathfrak{m}_x\tilde{M}_x \xrightarrow{\cdot 2} \tilde{N}_x/\mathfrak{m}_x\tilde{N}_x = (\mathbb{Z}/4\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z})$$

is the zero-map. This supplies a counter example, so the assertion is false.

¹See https://stacks.math.columbia.edu/tag/07RC (6)