Problem 1.

1. This is a local question and thus suffices to show on a cover of $|X|$. We pick a cover making both E and E' isomorphic to a direct summand of $\mathcal{O}_X$. Indeed, if we have such affine covers (V_j) and (V'_k) for E and E', respectively, we take the indicies s.t. $x \in V_j \cap V'_k$ and choose some affine open $x \in U_i \subset V_j \cap V'_k$. Then clearly both E and E' restricts to some direct summand of $\mathcal{O}_X$ on these U_i. Taking stalks, we find an exact sequence of $\mathcal{O}_{X,x}$-modules:
$$0 \longrightarrow \ker(\phi_{|U_i,x}) \longrightarrow \mathcal{O}_{X,x}^{\oplus m} \longrightarrow \mathcal{O}_{X,x}^{\oplus n} \longrightarrow 0.$$
Using that $\mathcal{O}_{X,x}^{\oplus n}$ is a projective $\mathcal{O}_{X,x}$-module, we find $\mathcal{O}_{X,x}^{\oplus m} \cong \mathcal{O}_{X,x}^{\oplus n} \oplus \ker(\phi_{|U_i,x})$ over $\mathcal{O}_{X,x}$. But it is a fact that if two R-modules M and N where R is a local ring, we have $M \oplus N \cong R^k$ implies $M \cong R^s$ and $N \cong R^t$ with $s + t = k$. Hence we find $\ker(\phi_{|U_i,x})$ is a direct summand of $\mathcal{O}_{X,x}$ as desired.

2. For a counter example, let k be a field and pick the ring $R = k[x]/(x^2)$ and let η be the map of vector bundles associated to the map $R \xrightarrow{\cdot x} R$ via. the $\tilde{(-)}$ construction. As $\ker(R \xrightarrow{\cdot x} R) = \{xr | r \in k\} \cong k$, we find that $\ker(\eta)$ correspond tot k, but k is clearly not a free R-module of finite rank.

3. The assertion is false. We pick $R = \mathbb{Z}$ and see that the map $R \xrightarrow{\cdot 2} R$ has kernel $\mathbb{Z}/2\mathbb{Z}$. We see that $\widetilde{\mathbb{Z}/2\mathbb{Z}}$ is not a vector bundle as $\mathbb{Z}/2\mathbb{Z}$ has torsion and thus isn't flat. But then applying $\tilde{(-)}$ to the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ of $\mathbb{Z}$-modules, we get an exact sequence $0 \to \tilde{\mathbb{Z}} \xrightarrow{\psi} \tilde{\mathbb{Z}} \to \widetilde{\mathbb{Z}/2\mathbb{Z}} \to 0$ and hence conclude that $\mathrm{coker}(\psi)$ is not a vector bundle.

Problem 4. First we note that from Problem 3 that $\mathcal{L}^{-1} \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ - the internal Hom sheaf. We also note that tensoring with an invertible sheaf (line bundle) preserves exactness (and thus reflects as we can tensor with the inverse line bundle), as it is locally trivial.

$(\Rightarrow)$: Assuming $\mathcal{L} \cong \mathcal{J} < \mathcal{O}_X$ for some sheaf of ideals $\mathcal{J}$, we find an injective inclusion map $\mathcal{L} \xrightarrow{i} \mathcal{O}_X$. This defines a non-zero global section of $\mathcal{L}^{-1}$ under the identification $\mathcal{L}^{-1} \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ (it is non-zero as the map is injective).

$(\Leftarrow)$: Assume $\mathcal{L}^{-1}$ has a non-zero global section $s \in \mathcal{L}^{-1}(|X|)$. Consider the closed subscheme $V = X \backslash X_s = \{x \in |X| : s(x) = 0 \text{ in } \mathcal{L}(x)\}$ of X. We claim that this is a Cartier divisor whose corresponding sheaf of ideals is isomorphic to $\mathcal{L}$, which finishes the problem.
Pick a trivializating cover $(U_i)_{i \in I}$ of $|X|$ and isomorphisms $\phi_i : \mathcal{L}_{|U_i} \xrightarrow{\sim} \mathcal{O}_{X|U_i}$. Let $f_i = \phi_i(s_{|U_i})$ which is a non-zero-divisor, as $\mathcal{O}_X(U_i) \cong A_i$ is an integral domain since X is an integral scheme. We claim that $V \cap U_i \cong \mathrm{Spec}(\mathcal{O}_X(U_i)/f_i)$. Indeed, when restricting to U_i, $X \backslash V$ is just a standard open due to the trivialization on U_i, so we see that $x \in V \cap U_i$ iff the corresponding prime ideal $\mathfrak{p}$ in A_i contains $f_i = \phi(s_{|U_i})$ (recall $s(x) = 0$ in U_i iff $s \cong f_i \in \mathfrak{p} = x$). But this equivalent to $x \in V((f_i)) \cong V(\ker(A_i \to A_i/f_i)) \cong \mathrm{Spec}(\mathcal{O}_X(U_i)/f_i)$.

Recall that the corresponding sheaf of ideals is the sheaf $\ker(\mathcal{O}_X \to \mathcal{O}_V)$. So we want to have an exact sequence of $\mathcal{O}_X$-modules
$$0 \to \mathcal{L} \to \mathcal{O}_X \to \mathcal{O}_V \to 0,$$
with first map is coming from tensoring the map $0 \to \mathcal{O} \xrightarrow{\cdot s} \mathcal{L}^{-1}$ with $\mathcal{L}$. It is exact at $\mathcal{O}_V$ as $V \xrightarrow{\iota} X$ is a closed immersion and thus $\mathcal{O} \to \iota_* \mathcal{O}_V = \mathcal{O}_V$ is surjective. Now, applying $- \otimes_{\mathcal{O}_X} \mathcal{L}^{-1}$, it is equivalent to show the following is exact:
$$0 \to \mathcal{O}_X \xrightarrow{\cdot s} \mathcal{L}^{-1} \xrightarrow{1 \otimes -} \mathcal{O}_V \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to 0,$$
Note that the first map is injective since s is non-zero and X is integral (i.e. we have cancelation of non-zeros and the map is multiplication). Exactness at the middle is a local question so suffices to

check on restriction to an open cover; we pick (U_i) from before. Given $x \in |X|$, pick open trivializing ngbh U_i of x. Using that $(\mathcal{O}_V \otimes_{\mathcal{O}_X} \mathcal{L}^{-1})_{|U_i} = \mathcal{O}_{V|U_i} \otimes_{\mathcal{O}_{X|U_i}} \mathcal{L}^{-1}_{|U_i}$ want the following to be exact:

$$\mathcal{O}_{X|U_i} \xrightarrow{\cdot s_{|U_i} \equiv \cdot f_i} \mathcal{L}^{-1}_{|U_i} \cong \mathcal{O}_{X|U_i} \longrightarrow \mathcal{O}_{V|U_i} \cong \mathcal{O}_{\mathrm{Spec}\,(A_i/f_i)}$$

But on stalks, this is just a consequence of exactness of the following exact sequence:

$$A_i \xrightarrow{\cdot f_i} A_i \longrightarrow A/f_i,$$

which is obviously exact.