Problem 1. Let $\mathbb{A}^1_k = \operatorname{Spec}(k[t])$ where $t = x^{-1}$. Consider the map $\operatorname{Spec}(k(x)) \to \mathbb{A}^1_k$ induced by the map $\phi : k[t] \to k(x)$, $t \mapsto x^{-1}$. Assuming that the map of schemes extends to $\operatorname{Spec}(R) = \operatorname{Spec}(k[x]_{(x)})$ we find a factorization $k[x^{-1}] \to k[x]_{(x)} \to k(x)$ which is impossible since x is not inverted in $k[x]_{(x)}$. Hence the map $\operatorname{Spec}(k(x)) \to \mathbb{A}^1_k$ does not extend to a map from $\operatorname{Spec}(R)$. Now R is an integral domain since k is a field. Also, any rational function with denominator not divisible by x lies in $k[x]_{(x)}$ by definition, and if p/q (written in lowest terms) has x in its denominator, its inverse q/p does not, and thus lies in $k[x]_{(x)}$. Hence $k[x]_{(x)} \subset k(x)$ is a valuation ring. So we have found a counter example to the valuative criterion for properness, and thus $\mathbb{A}^1_k$ cannot be proper over $\operatorname{Spec}(k)$.

$$\begin{array}{ccc} \operatorname{Spec}(k(x)) & \longrightarrow & \mathbb{A}^1_k \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec}(k[x]_{(x)}) & \longrightarrow & \operatorname{Spec}(k) \end{array} \qquad \begin{array}{ccc} k(x) & \xleftarrow{x^{-1} \leftarrow\!\shortmid t} & k[t] = k[x^{-1}] \\ \uparrow & \swarrow & \uparrow \\ k[x]_{(x)} & \hookleftarrow & k \end{array}$$

Problem 3.

1. Let $\mathcal{L}$ be a line bundle on $\mathbb{A}^1_k$. Then we have $\mathcal{L} \cong \tilde{L}$ for some finitely generated projective module L over $k[t]$. But since $k[t]$ is a PID (in fact, a EUD) we get L is trivial. I.e. $L \cong k[t]^{\oplus n}$. But $\mathcal{L}$ is locally free of rank 1 so $n = 1$. Thus $\mathcal{L} \cong \widetilde{k[t]} = \mathcal{O}_{\mathbb{A}^1_k}$.

2. Let $\mathcal{L}$ be a line bundle on $\mathbb{P}^1_k$. Then $\mathcal{L}_{|U_0}$ is a line bundle on $U_0 = \operatorname{Spec}(k[x]) \cong \mathbb{A}^1_k$, so from above, $\mathcal{L}_{|U_0} \cong \mathcal{O}_{\mathbb{A}^1_k} \cong \mathcal{O}_{\mathbb{P}^1_k |U_0}$. Similarly, $\mathcal{L}$ restricts trivially on U_1. But due to the equivalence of categories of line bundles trivial on (U_i) and descent datum on (U_i), giving $\mathcal{L}$ amounts to giving a unit $u \in \mathcal{O}_{\mathbb{P}^1_k}(U_0 \cap U_1)^\times \cong k[x, x^{-1}]^\times$. As the only invertibles in the ring $k[x, x^{-1}]$ are powers of x times an element from $k^\times$. If we have this, we see that $\mathcal{L} \cong \mathcal{O}(n)$ for some $n \in \mathbb{Z}$, as this was how the line bundles $\mathcal{O}(n)$ were defined.

$$\begin{array}{ccc} U_0 \cap U_1 = \operatorname{Spec}(k[x, x^{-1}]) & \xrightarrow{x \mapsto x} & \operatorname{Spec}(k[x]) = U_0 \\ {\scriptstyle y \mapsto x^{-1}} \downarrow & & \downarrow \\ U_1 = \operatorname{Spec}(k[y]) & \longrightarrow & \mathbb{P}^1_k \end{array}$$

Finally, we prove the assertion about invertibles in $k[x, x^{-1}]$. Every element in this ring can be written as a sum of a polynomial in x and one in x^{-1}. So if an element $p_1(x) + q_1(x^{-1})$ is invertible, we find $(p_1(x) + q_1(x^{-1}))(p_2(x) + q_2(x^{-1})) = 1$. Then find $1 = (p_1(x)x^m + q_1(x^{-1})x^m)(p_2(x)x^{-m} + q_2(x^{-1})x^{-m})$ for some sufficiently large power m so we can assume $1 = p(x)q(x^{-1})$ for polynomials p, q of same degree. Then, assume there is some term ax^i for $0 \leq i < n$. Then in the product $p(x)q(x^{-1})$, there is a term of the form ax^{i-n}. But this cannot happen as the product is 1 which is of degree 0. Hence p is monic, i.e. on the form ax^n for some $n \in \mathbb{Z}$.