Problem 6.

1. Let X be quasi-compact and $A \subset X$ closed. We show A is quasi-compact. Given an open cover $A = \bigcup_{i\in I} U_i$, then $X = A \cup (X\backslash A) = \bigcup_{i\in I} U_i \cup (X\backslash A)$ which is an open cover of X and thus admits a finite exhaustion; $X = \bigcup_{i\in J} U_i \cup (X\backslash A)$, $J \subset I$ finite ($X\backslash A$ is in this exhaustion as $A = \bigcup_{i\in I} U_i$ and $X\backslash A$ are disjoint). Hence $A = \bigcup_{i\in J} U_i$ is a finite cover.

2. From Corollary 1.9 we know that the standard opens form a basis. So we only prove that the prime spectrum of a ring is quasi-compact and the standard opens are so. Let $X = |\operatorname{Spec} R|$ be the prime spectrum of a ring. Suffices to show that a cover of standard opens exhausts a finite cover, as these opens form a basis. So given $X = \bigcup_{i\in I} D(f_i)$ for $f_i \in R$, we find $\emptyset = \bigcap_{i\in I} V(f_i) = V(\bigcup_{i\in I}(f_i)) = V(\sum_{i\in I}(f_i))$. Hence $1 = \sum_{i\in J} r_i f_i$ for some $r_i \in R$ and $J \subset I$ finite. Going back, we find $\emptyset = V(1) = V(\bigcup_{i\in J}(f_i)) = \bigcap_{i\in J} V(f_i)$ and taking complements, we get $|\operatorname{Spec}(R)| = \bigcup_{i\in J} D(f_i)$ as wanted.

   Next, we prove $D(f)$ is quasi-compact for all $f \in R$. Indeed, given a standard open cover $D(f) = \bigcup_{i\in I} D(g_i) = D((g_i)_{i\in I})$, we take complements and use Proposition 1.6 to find $\sqrt{(f)} = \sqrt{(g_i)_{i\in I}}$. Thus there are $n \geq 1$ and $r_i \in R$ s.t. $f^n = \sum_{i\in J} r_i g_i$ for some finite $J \subset I$. Then we find $\sqrt{(f)} = \sqrt{(f)^n} = \sqrt{(g_i)_{i\in J}}$ so $V(f) = V((g_i)_{i\in J})$, and thus $D(f) = \bigcup_{i\in J} D(g_i)$ which is a finite exhaust as we wanted.

3. Given some open $U \subset |\operatorname{Spec}(A)|$ s.t. $U^{\complement} = V(I)$ for some finitely generated ideal $I = (f_1, \dots, f_n)$ of A, we must show $U = V(I)^{\complement} = D(f_1, \dots, f_n)$ is quasi-compact. But this follows from the fact that a finite union of quasi-compacts is itself quasi-compact; we have $D(f_1, \dots, f_n) = \bigcup_{i=1}^n D(f_i)$ and above subquestion justify the quasi-compactness of $D(f_i)$.

   Conversely, take an open subset $U \subset |\operatorname{Spec}(A)|$ and assume it is quasi-compact. As the standard opens form a basis, we find $U = \bigcup_{i\in I} D(f_i)$ for some $f_i \in R$. Then we find a finite exhaust $U = \bigcup_{i\in J} D(f_i)$ for a finite subset $J \subset I$. But then $U^{\complement} = \bigcap_{i\in J} V(f_i) = V((f_i)_{i\in J})$ which is of the form $V(I)$ for some finitely generated ideal $I = (f_i)_{i\in J}$.

4. Let U_1 and U_2 be two opens which are quasi-compact. We must prove $U_1 \cap U_2$ is quasi-compact. By the previous subquestion, there are finitely generated ideals I and J s.t. ${U_1}^{\complement} = V(I)$ and ${U_2}^{\complement} = V(J)$. So given an open cover of standard opens; $U_1 \cap U_2 = \bigcup_{i\in I} D(f_i) = D((f_i)_{i\in I})$ we find $V((f_i)_{i\in I}) = (U_1 \cap U_2)^{\complement} = {U_1}^{\complement} \cup {U_2}^{\complement} = V(I) \cup V(J) = V(I \cdot J)$ by proposition 1.6. Now, as a product of finitely generated ideals is finitely generated itself, we conclude from the previous subqeustion that the open subset $U_1 \cap U_2$ is quasi-compact.

5. Given $x, y \in |\operatorname{Spec}(R)|$ different corresponding to the prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. As $x \neq y$ (and thus $\mathfrak{p} \neq \mathfrak{q}$), we can take $f \in \mathfrak{p}$ which is not in $\mathfrak{q}$. Consider

   $$V(f) = \{\mathfrak{r} \in |\operatorname{Spec}(R)| \big| f \in \mathfrak{r}\}.$$

   We see that $x = \mathfrak{p} \in V(f)$ and $y = \mathfrak{q} \notin V(f)$. As $V(f)$ is closed by definition, we conclude that $|\operatorname{Spec}(R)|$ is T_0.